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# Judgment aggregation and agenda manipulation

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**Abstract:** When individual judgments (‘yes’ or ‘no’) on some propositions are aggregated into collective judgments, outcomes may be sensitive to the choice of propositions under consideration (the *agenda*). Such agenda-sensitivity opens the door to manipulation by agenda setters. I define three types of agenda-insensitivity (‘basic’, ‘full’, and ‘focal’) and for each type axiomatically characterize the aggregation procedures satisfying it. Two axioms turn out to be central for agenda-insensitivity: the familiar *independence* axiom, requiring propositionwise aggregation, and the axiom of *implicit consensus preservation*, requiring the respect of any (possibly implicit) consensus. As the paper’s second contribution, I prove a new impossibility theorem whereby these two axioms imply dictatorial aggregation for almost all agendas. *JEL Class.:* D70, D71.

**Keywords:** judgment aggregation, multiple issues, description-sensitivity, agenda manipulation, impossibility theorems, characterization theorems

## 1 Introduction

Imagine that the board of a central bank has to form collective judgments (‘yes’ or ‘no’) on some propositions about the economy, such as the proposition that prices will rise. Disagreements on a proposition are resolved by taking a majority vote. The chair of the board knows that a majority believes prices won’t rise. Nonetheless he wants the board to form a collective judgment that prices will rise.<sup>2</sup> To achieve this goal, he removes the proposition ‘prices will rise’ from the agenda, while putting two new propositions on the agenda: ‘GDP will grow’, and ‘growth implies inflation’, i.e., ‘*if* GDP will grow, *then* prices will rise’. Once it comes to voting, the two new propositions are each approved by

	initial agenda	manipulated agenda	
	Inflation?	Growth?	Growth implies inflation?
Member 1	Yes	Yes	Yes
Member 2	No	No	Yes
Member 3	No	Yes	No
Majority	No	Yes	Yes

Figure 1: An agenda manipulation reversing the collective judgment on inflation

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<sup>2</sup>The reason might be his belief in imminent inflation, or his desire for the bank to raise interest rates (which happens only if the board concludes that there is an inflation risk). In the first case he cares about the truth of collective judgments. In the second case he cares about consequences (actions) resulting from collective judgments. This paper leaves open the motivation of agenda setters.

a (different) majority. The chair is pleased, since the collective beliefs in growth and in growth implying inflation logically entail a belief in inflation. This agenda manipulation has successfully turned an (explicit) ‘no inflation’ judgment into an (implicit) ‘inflation’ judgment. Figure 1 illustrates this reversal in the case of a three-member board.

This example shows that majority voting is vulnerable to agenda manipulation. Which rules (if any) are immune to agenda manipulation? This paper defines different types of agenda sensitivity, and characterizes the aggregation rules immune to each type. Two axioms on the aggregation rule turn out to play key roles in ensuring manipulation-immunity: *independence* (i.e., the analogue for judgment aggregation of Arrow’s axiom of *independence of irrelevant alternatives* for preference aggregation), and *implicit consensus preservation* (i.e., the principle of respecting unanimity, in a strengthened version extended to implicit judgments). In a new impossibility theorem, I prove that these two axioms can almost never be satisfied by an aggregation rule which is non-dictatorial (as well as having an unrestricted domain and generating rational collective judgments). This impossibility theorem is also of interest in its own right, i.e., independently of the issue of agenda manipulation. Indeed the two axioms need not be motivated by considerations of agenda manipulation. The paper therefore has two main contributions: an analysis of agenda manipulation, and the proof of a new impossibility theorem.

The present analysis of agenda sensitivity fills a gap in the literature on judgment aggregation, in which agenda sensitivity/manipulation is often mentioned informally and was treated in a semi-formal way by Dietrich (2006).<sup>3</sup> Other types of manipulation have however been much studied. One type is the manipulation of the aggregation rule, more precisely of the order of priority in which a sequential aggregation rule considers the propositions in the agenda (List 2004, Dietrich and List 2007c, Nehring, Pivato and Puppe 2014). Another type of manipulation is strategic voting, in which voters do not report truthfully their judgments. Strategic voting has been studied using two different approaches. One approach focuses on *opportunities* to manipulate, setting aside the behavioural question of whether voters take these opportunities or vote truthfully (e.g., Dietrich and List 2007b, Dokow and Falik 2012). The other approach focuses on *incentives* to manipulate, i.e., on actual voting behaviour (e.g., Dietrich and List 2007b, Dokow and Falik 2012, Ahn and Oliveros 2014, Bozbay, Dietrich and Peters 2014, DeClippel and Eliaz 2015; see also Nehring and Puppe 2002). The first approach requires only a basic, preference-free judgment-aggregation setup, whereas the second approach requires modelling voters’ preferences (and their private information, if any). The present paper studies whether an agenda setter has *opportunities* to manipulate via the choice of agenda. I leave open whether he is himself a voter or an external person, and whether he takes such opportunities or refrains from manipulation. The latter question depends on his preferences, which are not modelled here. Although manipulation behaviour is not addressed explicitly, it is overly clear that manipulation *opportunities* will lead to manipulation *behaviour* under many plausible preferential assumptions.<sup>4</sup>

The paper’s second contribution – a new impossibility theorem – connects to a series of impossibility results in the field; see for instance List and Pettit (2002), Pauly and van

<sup>3</sup>The limited overlap of the present paper with Dietrich (2006) is explained in Section 4.

<sup>4</sup>One such assumption is that the agenda setter holds preferences over outcomes that are totally independent of votes and voters’ information, as in our introductory example where the agenda setter simply wants a collective judgment of rising prices.

Hees (2006), Dietrich (2006), Dietrich and List (2007a), Mongin (2008), Nehring and Puppe (2008), Duddy and Piggins (2013), and papers in the *Symposium on Judgment Aggregation* in Journal of Economic Theory (C. List and B. Polak eds., 2010). Of particular interest to us is a theorem which generalizes Arrow’s Theorem from preference to judgment aggregation (Dietrich and List 2007a and Dokow and Holzman 2010, both building on Nehring and Puppe 2010 and strengthening Wilson 1975). The new theorem shows that if in the generalized Arrow theorem the Pareto-type unanimity condition is extended towards implicit agreements, then, perhaps surprisingly, the dictatorship conclusion now holds for *almost all* agendas, not just agendas of a quite special structure.

I should mention a growing branch of the literature which constructs concrete judgment aggregation rules, and whose attention I hope to draw to agenda manipulation. Many proposals have been made. Our analysis will imply that almost all proposals are vulnerable to agenda manipulation, yet in different ways and to different degrees.<sup>5</sup>

The paper is structured as follows. Section 2 defines the framework. Section 3 states and explains the impossibility theorem on propositionwise and implicit consensus preserving aggregation. Sections 4 and 5 address agenda-sensitivity, stating characterization and impossibility results. Section 6 adds concluding remarks. Appendix A defines an alternative framework (more typical for judgment-aggregation theory) in which all our results continue to hold. Appendix B contains all proofs.

## 2 The framework

I now define the judgment-aggregation framework (e.g., List and Pettit 2002 and Dietrich 2007, 2014). I define it in a *semantic* version, which takes propositions to be sets of possible worlds (‘events’) rather than abstract or syntactic objects. The semantic way of thinking is uncommon in the field, but familiar elsewhere in economics, and convenient in this paper.<sup>6</sup> But nothing hinges on using this framework: all formal results in the main text continue to hold in a general framework which is defined in Appendix A.

A group of  $n$  individuals, labelled  $i = 1, \dots, n$ , needs to form yes/no judgments on some interconnected propositions. We assume that  $n \geq 3$ .<sup>7</sup>

**The agenda.** Let  $\Omega$  be a fixed non-empty set of possible *worlds* or *states*. A *proposition* or *event* is a subset  $p \subseteq \Omega$ ; its *negation* or *complement* is denoted  $\bar{p} := \Omega \setminus p$ . Those propositions on which judgments (‘yes’ or ‘no’) are formed make up the *agenda*. As usual, I assume that the agenda is a union of pairs  $\{p, \bar{p}\}$ , the *issues* on the agenda. A board of a central bank might deal with the issues {growth, no-growth}, {inflation, no-inflation}, and so on. Formally:

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<sup>5</sup>The proposals include premise- and conclusion-based rules (e.g., Kornhauser and Sager 1986, List and Pettit 2002, Dietrich 2006, Dietrich and Mongin 2010), sequential rules (e.g., List 2004, Dietrich and List 2007b), distance-based rules (e.g., Konieczny and Pino-Perez 2002, Pigozzi 2006, Miller and Osherson 2009, Eckert and Klamler 2009, Lang *et al.* 2011, Duddy and Piggins 2012), quota rules with well-calibrated acceptance thresholds and various degrees of collective rationality (e.g., Dietrich and List 2007b; see also Nehring and Puppe 2010), aggregation rules for restricted domains (Dietrich and List 2010, Pivato 2009), relevance-based aggregation rules (Dietrich 2015), Borda-like and scoring rules (Dietrich 2014, Duddy, Piggins and Zwicker 2016), and rules which approximate the majority judgment set when it is inconsistent (Nehring, Pivato and Puppe 2014).

<sup>6</sup>The notion of the ‘scope’ of an agenda becomes more concrete.

<sup>7</sup>All theorems except the ‘only if’ part of Theorems 1 and 5 even hold for  $n \geq 2$ .

**Definition 1** An **agenda** is a proposition set  $X \subseteq 2^\Omega$  which is closed under negation, i.e.,  $p \in X \Leftrightarrow \bar{p} \in X$ , and which (in this paper) is finite and contains at least one proposition  $p \neq \Omega, \emptyset$ . Each pair  $\{p, \bar{p}\} \subseteq X$  is an **issue** of the agenda.<sup>8</sup>

The closure under negation of a proposition set  $Y$  is denoted  $Y^\pm := \bigcup_{p \in Y} \{p, \bar{p}\}$ . So I can conveniently write an  $m$ -issue agenda as  $X = \{p_1, \dots, p_m\}^\pm$ , where  $p_i$  belongs to the  $i^{\text{th}}$  issue. An individual's *judgment set* is the set of propositions in  $X$  he believes. The following are standard requirements on judgment sets:

**Definition 2** Given an agenda  $X$ , a judgment set  $J \subseteq X$  is **consistent** if  $\bigcap_{p \in J} p \neq \emptyset$ , **complete** if it contains a member of each issue  $\{p, \bar{p}\} \subseteq X$ , and **rational** if it is both consistent and complete. The set of rational judgment sets is denoted  $\mathcal{J}_X$  or just  $\mathcal{J}$ .

As a concrete example, assume  $\Omega = \{0, 1\}^3$ . In a world  $(j, k, l) \in \Omega$  the first component  $j$  indicates whether it is sunny (1) or not (0), the second one  $k$  whether it is warm (1) or not (0), and the third one  $l$  whether it is windy (1) or not (0). Consider the propositions  $p = \{(j, k, l) \in \Omega : j = 1\}$  (it's sunny),  $q = \{(j, k, l) \in \Omega : k = 1\}$  (it's warm) and  $r = \{(j, k, l) \in \Omega : l = 1\}$  (it's windy). Here are some potential agendas:

$$X = \{p, q, r\}^\pm, \quad X = \{p \cap q, p \cap r, q \cap r\}^\pm, \quad X = \{p, p \cup q, p \cup r\}^\pm. \quad (1)$$

The first of these agendas has no logical interconnections between its issues: all  $2^3 = 8$  judgment sets consisting of one proposition from each issue (i.e.,  $\{p, q, r\}$ ,  $\{\bar{p}, q, r\}$ ,  $\{p, \bar{q}, r\}$ ,  $\{\bar{p}, \bar{q}, r\}$ , ...) are consistent, hence in  $\mathcal{J}$ . The other two agendas have interconnected issues. For instance, the judgment set  $\{p \cap q, p \cap r, \overline{q \cap r}\}$  is inconsistent, hence not in  $\mathcal{J}$ .

**The scope of the agenda.** A judgment set  $J \subseteq X$  typically settles many more propositions than those it explicitly contains, where 'settling a proposition' means entailing whether it is true or false. For instance, although the first agenda in (1) does not contain the proposition  $p \cap q$ , this proposition is settled by the judgments on  $p$  and on  $q$ . In fact, for the first agenda any judgment set  $J \in \mathcal{J}$  settles *all* propositions  $p \subseteq \Omega$ ; I shall say that all propositions are in the agenda's *scope*. By contrast, for the other two agendas in (1) some propositions  $p \subseteq \Omega$  may remain unsettled, i.e., are out of the agenda's scope. The following definitions make all this precise.

**Definition 3** A proposition  $p$  (or proposition set  $J$ ) **entails** a proposition  $p'$  (or proposition set  $J'$ ) if  $p$  (resp.  $\bigcap_{q \in J} q$ ) is a subset of  $p'$  (resp.  $\bigcap_{q \in J'} q$ ).

**Definition 4** (Dietrich 2006) A proposition set  $J$  **settles** a proposition  $p$  if it entails  $p$  or entails  $\bar{p}$ . The **scope** of an agenda  $X$  is the set  $\bar{X}$  of propositions settled by each rational judgment set  $J \in \mathcal{J}_X$ ; equivalently, it is the closure of  $X$  under union (or intersection) and negation, i.e., the algebra generated by  $X$ .

<sup>8</sup>The finiteness restriction could be dropped in many results, e.g., those of Section 4.

The scope of an agenda can be quite large. It evidently contains all agenda propositions and all unions and intersections of agenda propositions. We can continue: it contains all negations of unions of agenda propositions, all intersections of negations of unions of agenda propositions, and so on. In short, the scope contains all propositions constructible from agenda propositions. For instance the scope of the first agenda in (1) contains all propositions:  $\overline{X} = 2^\Omega$ . Indeed, each rational judgment set uniquely determines a single world (e.g.,  $\{p, q, \bar{r}\}$  determines the world  $(1, 1, 0)$ ), hence is able to settle every proposition. Note that the scope  $\overline{X}$  of an agenda  $X$  is itself an agenda, where  $X \subseteq \overline{X}$ .

**Definition 5** *Two agendas  $X$  and  $X'$  are **equivalent** if they have same scope  $\overline{X} = \overline{X'}$ .*

For instance agendas  $X = \{p, q\}^\pm$  and  $X = \{p \cap q, \bar{p} \cap q, p \cap \bar{q}, \bar{p} \cap \bar{q}\}^\pm$  have the same scope and are thus equivalent. Equivalent agendas represent essentially the same decision problem, but framed differently.

Note that for any agenda  $X$  the *atoms* of the scope  $\overline{X}$  (the minimal non-empty propositions in  $\overline{X}$ ) are the intersections of maximally many mutually consistent propositions in  $X$ , i.e., the propositions  $\bigcap_{p \in J} p$  where  $J \in \mathcal{J}_X$ .

**Aggregation rules.** An **aggregation rule** for an agenda  $X$  is a function  $F$  which to every profile of ‘individual’ judgment sets  $(J_1, \dots, J_n)$  (from some domain, usually  $\mathcal{J}^n$ ) assigns a ‘collective’ judgment set  $F(J_1, \dots, J_n)$ . For instance, majority rule is given by

$$F(J_1, \dots, J_n) = \{p \in X : \text{more than half of } J_1, \dots, J_n \text{ contain } p\}$$

and generates inconsistent collective judgment sets for many agendas and profiles. We shall be concerned with aggregation rules whose individual inputs and collective output are rational. Such rules are functions  $F : \mathcal{J}^n \rightarrow \mathcal{J}$ . Note that we exclude ties in this paper: our aggregation rules are by definition ‘resolute’.

**The example of preference aggregation.** For a set  $A$  of ‘alternatives’ with  $2 \leq |A| < \infty$ , let  $\Omega$  be the set of strict linear orders  $\succ$  on  $A$ , where  $x \succ y$  reads ‘ $x$  is better than  $y$ ’ according to a given (objective) criterion. So worlds describe how the alternatives are (objectively) ranked. The group disagrees on the ranking. The **preference agenda** is defined as  $X = \{xPy : x, y \in A, x \neq y\}$ , where  $xPy$  is the proposition that  $x$  is better than  $y$ , i.e.,  $xPy = \{\succ \in \Omega : x \succ y\}$  (note that  $\overline{xPy} = yPx$ ). There is a one-to-one correspondence between rational judgment sets  $J \in \mathcal{J}$  and strict linear orders  $\succ$  on  $A$ , given by  $xPy \in J \Leftrightarrow x \succ y$ . Aggregation rules  $F : \mathcal{J}^n \rightarrow \mathcal{J}$  can thus be regarded as preference aggregation rules. Aside from this formal analogy between preference aggregation and judgment aggregation for the preference agenda, there is an interpretational difference: preferences are usually viewed as attitudes of comparative desire, not judgments (beliefs) about an objective ranking.

### 3 The impossibility of implicit consensus preserving propositionwise aggregation

I now state two axioms on an aggregation rule  $F : \mathcal{J}^n \rightarrow \mathcal{J}$  for a given agenda  $X$ ; they will jointly lead to an impossibility result. Each axiom is interesting in itself, but also

matters ‘instrumentally’ by helping to limit agenda manipulation, as will be shown in depth in Section 4.

The first axiom is the classical condition of ‘independence’ or ‘propositionwise aggregation’. It requires the collective judgment on any given proposition in the agenda to depend solely on the individuals’ judgments on *this* proposition – the judgment-aggregation analogue of Arrow’s axiom of *independence of irrelevant alternatives* (to which it reduces in the case of the preference agenda).

**Independence (‘propositionwise aggregation’):** For all propositions  $p \in X$  and profiles  $(J_1, \dots, J_n), (J'_1, \dots, J'_n) \in \mathcal{J}^n$ , if  $p \in J_i \Leftrightarrow p \in J'_i$  for every individual  $i$ , then  $p \in F(J_1, \dots, J_n) \Leftrightarrow p \in F(J'_1, \dots, J'_n)$ .

This axiom is normatively no less controversial than Arrow’s analogous axiom. It is known to be necessary for preventing strategic voting (Dietrich and List 2007b). We here focus on its role in preventing agenda manipulation. As shown in Section 5, it is also necessary for preventing an agenda manipulator from being able to reverse explicit collective judgments. In short, if independence is violated, then the collective judgment on a proposition  $p \in X$  depends on other propositions in the agenda, and can thus be reversed by the agenda setter through adding or removing other propositions.

Our second axiom requires respecting consensus, in an unusually strong sense. I first recall the two standard consensus conditions, which pertain to judgment-set-wise resp. proposition-wise consensus:

**Unanimity preservation:**  $F(J, \dots, J) = J$  for each unanimous profile  $(J, \dots, J) \in \mathcal{J}^n$ .

**Unanimity principle:** For all  $(J_1, \dots, J_n) \in \mathcal{J}^n$  and  $p \in X$ , if each  $J_i$  contains  $p$ , so does  $F(J_1, \dots, J_n)$ .

The first of these axioms is weaker and almost unobjectionable. The second one is analogous to the Pareto principle (and equivalent to it for the preference agenda). Our own consensus axiom resembles the latter axiom, but strengthens it by also covering ‘implicit’ consensus on propositions outside the agenda. The axiom can be stated in three equivalent versions.

**Implicit consensus preservation (version 1):** For every proposition  $p$  in the agenda’s scope  $\overline{X}$ , if each judgment set in a profile  $(J_1, \dots, J_n) \in \mathcal{J}^n$  entails  $p$ , then  $F(J_1, \dots, J_n)$  entails  $p$ .

This axiom is demanding. It for instance implies that whenever every individual accepts at least one of some given propositions in  $X$ , i.e., implicitly endorses their disjunction (union), then so does the collective – which might conflict with majority voting since each of these propositions might be rejected by a majority. In the case of the preference agenda, the axiom for instance implies that if every individual ranks a certain alternative  $x$  in 2<sup>nd</sup> position, i.e., implicitly endorses the proposition ‘ $x$  is the 2<sup>nd</sup> best alternative’, then so does the collective – although many standard preference aggregation rules (such as Borda rule) sometimes rank in 1<sup>st</sup> position an alternative which everyone ranks in 2<sup>nd</sup> position.

Later I give two formal arguments for this axiom, both related to the prevention of agenda manipulation. Let me anticipate them very briefly. Firstly, the axiom prevents a particularly bad form of agenda sensitivity, in which unanimously supported collective judgments, explicit or implicit ones, are being reversed (see Section 4). Secondly, the axiom is effectively insensitive to redescribing (‘reframing’) the decision problem: the set of propositions  $p$  on which consensus must be preserved stays the same if the agenda  $X$  is replaced by a new one which has the same scope and is thereby equivalent (see Section 6).<sup>9</sup>

The axiom can be reformulated using the notion of a *feature* of a judgment set. Examples are the feature of containing a given proposition  $p \in X$ , and the feature of containing at most two propositions from a given set  $S \subseteq X$ . We may identify each feature with the set  $\mathcal{K} \subseteq \mathcal{J}$  of judgment sets having the feature. In its second version, our axiom requires the collective judgment set to have each feature shared by all individual judgment sets:

**Implicit consensus preservation (version 2):** For every  $\mathcal{K} \subseteq \mathcal{J}$  (every feature), if each judgment set in a profile  $(J_1, \dots, J_n) \in \mathcal{J}^n$  belongs to  $\mathcal{K}$  (has the feature), so does the collective judgment set  $F(J_1, \dots, J_n)$ .

Intuitively, the versions 1 and 2 are equivalent because a judgment set  $J \in \mathcal{J}$  has a given feature just in case it entails a certain proposition from the scope. For instance,  $J$  contains two given propositions  $q$  and  $r$  from  $X$  just in case it entails the proposition  $q \cap r$  from  $\overline{X}$ . In its third version, the axiom requires the collective judgment set to be selected from the set of individual judgment sets:

**Implicit consensus preservation (version 3):** For every profile  $(J_1, \dots, J_n) \in \mathcal{J}^n$ , the collective judgment set  $F(J_1, \dots, J_n)$  belongs to  $\{J_1, \dots, J_n\}$ .

This axiom is far from an (undemocratic) dictatorship requirement, since the individual whose judgment set becomes the collective one may vary with the profile; he could for instance be the profile’s ‘median’ voter in a suitably defined sense.

**Proposition 1** *The three versions of implicit consensus preservation are equivalent.*

I now combine our two axioms into an impossibility result. An aggregation rule  $F : \mathcal{J}^n \rightarrow \mathcal{J}$  is **dictatorial** if there is an individual  $i$  such that  $F(J_1, \dots, J_n) = J_i$  for all  $J_1, \dots, J_n \in \mathcal{J}$ . As usual in the theory, the structure of the agenda matters. The agenda  $X$  is called **nested** if it takes the very special form  $X = \{p_1, p_2, \dots, p_m\}^\pm$  where  $m$  is the number of issues and  $p_1 \subseteq p_2 \subseteq \dots \subseteq p_m$  (whence also  $\overline{p}_m \subseteq \overline{p}_{m-1} \subseteq \dots \subseteq \overline{p}_1$ ). For instance, the board of a bank might face such a nested agenda where  $p_j$  is the proposition ‘prices will grow by at most  $j$  percent’; and an academic hiring committee might face such a nested agenda where  $p_j$  is the proposition ‘candidate Smith will publish fewer than  $j$  papers per year’. But most relevant agendas are not nested. The agendas in (1) are not nested, and also the preference agenda defined in Section 2 is not nested (as long as there are more than two alternatives). Finally, the agenda  $X$  is **tiny** if it has at most two issues  $\{p, \overline{p}\}$  ( $\neq \{\Omega, \emptyset\}$ ), i.e., at most four propositions ( $\neq \Omega, \emptyset$ ).

<sup>9</sup>I thank Marcus Pivato for bringing this fact to my attention.



**Theorem 1** *Given an agenda, all independent and implicit consensus preserving aggregation rules  $F : \mathcal{J}^n \rightarrow \mathcal{J}$  are dictatorial if and only if the agenda is non-nested and non-tiny.*

To paraphrase the result, for almost all agendas our two axioms cannot be jointly satisfied by any non-dictatorial aggregation rule. Indeed, far more agendas imply impossibility than in the Arrow-like theorem mentioned in the introduction (and formally stated later as Theorem 5). For instance all agendas in (1) fall under the impossibility of Theorem 1 (they are non-nested), but not under that of the Arrow-like theorem. The same is true of almost all example agendas used repeatedly in the literature to illustrate inconsistent majority judgments, such as agendas of type  $\{p, q, p \cap q\}^\pm$ . Theorem 1's very wide class of 'impossibility agendas' is a result of requiring implicit consensus preservation, while standard impossibility theorems usually require one of the two weaker consensus axioms mentioned earlier.

Theorem 1's 'only if' part is established by showing that, for a nested agenda  $X = \{p_1, \dots, p_m\}^\pm$  (where  $p_1 \subseteq p_2 \subseteq \dots \subseteq p_m$ ), propositionwise majority rule satisfies all requirements.<sup>10</sup> In short, this is because each rational judgment set in  $\mathcal{J}$  takes the special form  $\{\overline{p_1}, \dots, \overline{p_{k-1}}, p_k, \dots, p_m\}$  for some cut off point  $k$  (in  $\{1, \dots, m+1\}$ ), and propositionwise majority rule returns the judgment set of an individual who is *median* in terms of the cut-off point. This argument has the flavour of single-peakedness and other structural conditions in preference or judgment aggregation. More precisely, nested agendas have the special property that all profiles in  $\mathcal{J}^n$  automatically satisfy several structural conditions which guarantee consistent majority judgments, i.e., all conditions introduced in List (2003) or Dietrich and List (2010).<sup>11</sup> Judgment-aggregation theorists will also be curious whether the notion of a non-nested agenda is related to any familiar kind of agenda. Non-nested agendas can in fact be related to *non-simple* agendas.<sup>12</sup>

<sup>10</sup>If  $n$  is even, then the majority is taken among individuals  $1, \dots, n-1$  only, to avoid ties.

<sup>11</sup>Consider, say, the condition of *unidimensional alignment* (a judgment-aggregation variant of *single-crossingness* and Rothstein's 1990 *order restriction*, to which it reduces if  $X$  is the preference agenda). A profile  $(A_1, \dots, A_n)$  is unidimensionally aligned if the individuals can be linearly ordered such that, for each proposition  $p \in X$ , the individuals  $i$  with  $p \in A_i$  all come before or all come after those with  $p \notin A_i$ . The order might represent a political left-right order, with the individuals accepting a 'left-wing' proposition  $p$  located to the left of those rejecting it. For nested  $X$ , all profiles in  $\mathcal{J}^n$  are necessarily unidimensionally aligned: just order the individuals by increasing cut-off point (the ranking between two individuals with same cut-off point can be chosen arbitrarily).

<sup>12</sup>An agenda  $X$  is **simple** if it has no subset  $Y \subseteq X$  with  $|Y| > 2$  that is *minimal inconsistent*, i.e., is inconsistent but becomes consistent if any member is removed (informally, simplicity means that there are no 'complex' interconnections involving more than two propositions). For instance, the preference agenda for a set of more than two alternatives is non-simple, since any 'cyclical' subset  $Y = \{xPy, yPz, zPx\}$  is minimal inconsistent. I show in Appendix B that a (non-tiny) agenda  $X$  is nested *if and only if* it satisfies a condition only subtly distinct from the definition of simplicity:  $X$  has no subset  $Y$  with  $|Y| > 2$  such that  $(Y \setminus \{p\}) \cup \{\overline{p}\}$  is consistent for each  $p \in Y$ . Adding 'inconsistent' before 'subset  $Y$ ' turns this characterization of nestedness into one of simplicity. This gives an idea of *how* nestedness strengthens simplicity.

## 4 Three types of agenda-insensitivity and their characterizations

I now define and characterize three forms of insensitivity of outcomes to the agenda choice (and hence, to agenda manipulation). The characterization results show that our two axioms – independence and implicit consensus preservation – play key roles in ensuring agenda-insensitivity, along with other axioms.

Think of the agenda  $X$  as being chosen by an agenda setter. This agenda setter has some room for maneuver, i.e., some degree of freedom in designing the agenda. Typically his influence has limits: he might be able to ‘reframe’ the decision problem, but not to alter its topic altogether. For instance he cannot remove all financial issues from the agenda of a bank’s board. The agenda setter might also face restrictions on the agenda’s complexity or size: perhaps he cannot set an agenda with more than three issues. To capture that only certain agendas  $X \subseteq 2^\Omega$  are feasible (choosable, settable), we consider a fixed set  $\mathcal{X}$  of agendas  $X \subseteq 2^\Omega$  deemed feasible/possible. It could consist of *all* agendas  $X \subseteq 2^\Omega$ ; or of all agendas with at most six issues; or of all agendas *without* certain given issues (the ‘too complex’ issues, say); and so on. All we require from  $\mathcal{X}$  is that it contains each single-issue agenda  $\{p, \bar{p}\} \subseteq \bigcup_{X \in \mathcal{X}} X$ . In particular,  $\mathcal{X}$  need not

contain unions  $X \cup X'$  of agendas  $X, X' \in \mathcal{X}$ , the scope  $\overline{X}$  of agendas  $X \in \mathcal{X}$ , or the maximal agenda  $X = 2^\Omega$ .

Can the agenda setter reverse collective judgments by changing the agenda? This question obviously depends on which aggregation rules would be used for the various feasible agendas. That is, it depends on what I call the ‘aggregation system’:

**Definition 6** An **aggregation system** is a family  $(F_X)_{X \in \mathcal{X}}$  containing an aggregation rule  $F_X : \mathcal{J}_X^n \rightarrow \mathcal{J}_X$  for each feasible agenda  $X \in \mathcal{X}$  (where  $F_X$  represents the rule used if the agenda is  $X \in \mathcal{X}$ ).<sup>13</sup>

I now define three conditions on an aggregation system  $(F_X)_{X \in \mathcal{X}}$ . Each one requires the outcomes to be in a specific sense insensitive to the agenda choice, hence, immune to agenda manipulation. The first condition states that the agenda setter cannot reverse any *explicit* collective judgment, i.e., any collective judgment on a proposition *in* the agenda:

**Definition 7** An aggregation system  $(F_X)_{X \in \mathcal{X}}$  is **basically agenda-insensitive** – for short, **agenda-insensitive** – if any two feasible agendas  $X, X' \in \mathcal{X}$  lead to the same collective judgment on any proposition  $p \in X \cap X'$ : for all  $J_1, \dots, J_n \in \mathcal{J}_{X \cup X'}$ ,

$$p \in F_X(J_1 \cap X, \dots, J_n \cap X) \Leftrightarrow p \in F_{X'}(J_1 \cap X', \dots, J_n \cap X').$$

What is the rationale behind this axiom? Think of  $J_i$  as individual  $i$ ’s judgment set under the (hypothetical) agenda  $X \cup X'$ , and think of  $J_i \cap X$  and  $J_i \cap X'$  as his submitted judgment sets under the (feasible) agendas  $X$  and  $X'$ , respectively. Note the implicit

<sup>13</sup>An aggregation system could be viewed as a single ‘extended aggregation rule’ with an additional argument, the agenda. Note that each rule  $F_X$  must have domain  $\mathcal{J}_X^n$  and co-domain  $\mathcal{J}_X$  (this restriction might be lifted by a more general definition of ‘aggregation system’).

idea that individuals hold fixed, i.e., agenda-independent, judgments on all propositions  $p \subseteq \Omega$ . In short, individuals are themselves agenda-insensitive in their judgments. A failure of individual agenda-insensitivity would of course open up additional sources of agenda manipulation, which we do not model here.

By the next theorem, agenda-insensitivity forces each rule  $F_X$  to be independent. It also forces the aggregation rule  $F_X$  to change *coherently* when the agenda setter extends the agenda  $X$  to a new agenda  $X'$ . What do I mean exactly? I start with two obvious definitions:

**Definition 8** *An agenda  $X'$  **extends** another one  $X$  if  $X \subseteq X'$ .*

**Definition 9** *A set of propositions  $A$  is **consistent with** another one  $B$  if  $A \cup B$  is consistent.*

I can now formally define what it means for the aggregation rule to change coherently as the agenda changes. For future purposes, the definition is formulated in full generality, i.e., for arbitrary agenda changes, not just agenda *extensions*:

**Definition 10** *In an aggregation system  $(F_X)_{X \in \mathcal{X}}$ , a rule  $F_{X'}$  **coheres with** a rule  $F_X$  if the outcome of  $F_{X'}$  is not ruled out by that of  $F_X$ : for any any  $J_1, \dots, J_n \in \mathcal{J}_X$ ,  $F_{X'}(J'_1, \dots, J'_n)$  is consistent with  $F_X(J_1, \dots, J_n)$  for at least some  $J'_1, \dots, J'_n \in \mathcal{J}_{X'}$  consistent with  $J_1, \dots, J_n$ , respectively.*

When do we call an entire aggregation system ‘coherent’?

**Definition 11** *An aggregation system  $(F_X)_{X \in \mathcal{X}}$  is **coherent** if whenever an agenda  $X \in \mathcal{X}$  is extended to another  $X' \in \mathcal{X}$  the rule  $F_{X'}$  coheres with  $F_X$ .*

The following remark gives a clear idea of what it means for  $F_{X'}$  to cohere with  $F_X$  as the agenda is extended:

**Remark 1** *In case  $X'$  extends  $X$ , coherence of  $F_{X'}$  with  $F_X$  means that the outcome of  $F_{X'}$  extends that of  $F_X$  for at least some extension of the individual judgments: for any  $J_1, \dots, J_n \in \mathcal{J}_X$ ,  $F_{X'}(J'_1, \dots, J'_n)$  extends  $F_X(J_1, \dots, J_n)$  for at least some  $J'_1, \dots, J'_n \in \mathcal{J}_{X'}$  extending  $J_1, \dots, J_n$ , respectively.*

I can now state the characterization result about agenda-insensitivity.

**Convention:** *For any property of aggregation rules (such as independence), an aggregation system  $(F_X)_{X \in \mathcal{X}}$  is said to satisfy it if and only if each rule  $F_X$  satisfies it.*

**Theorem 2** *An aggregation system  $(F_X)_{X \in \mathcal{X}}$  is agenda-insensitive if and only if it is independent and coherent.*

Basic agenda-insensitivity only prevents the agenda setter from reversing *explicit* collective judgments, on proposition *in* the agenda. We now turn to a stronger requirement, which also excludes the reversal of *implicit* collective judgments, on propositions

outside the agenda. For instance, if an agenda  $X = \{p, q\}^\pm$  leads the collective judgment set  $\{p, \bar{q}\}$ , so that the collective implicitly accepts the proposition  $p \cap \bar{q}$  from the scope  $\bar{X}$ , then the acceptance of  $p \cap \bar{q}$  cannot be reversed by using another agenda  $X'$ . Formally:

**Definition 12** *An aggregation system  $(F_X)_{X \in \mathcal{X}}$  is **fully agenda-insensitive** if any two feasible agendas  $X, X' \in \mathcal{X}$  lead to the same collective judgment on any proposition  $p \in \bar{X} \cap \bar{X}'$ : for all  $J_1, \dots, J_n \in \mathcal{J}_{X \cup X'}$ ,*

$$F_X(J_1 \cap X, \dots, J_n \cap X) \text{ entails } p \Leftrightarrow F_{X'}(J_1 \cap X', \dots, J_n \cap X') \text{ entails } p.$$

Here,  $J_i$ ,  $J_i \cap X$  and  $J_i \cap X'$  again represents individual  $i$ 's judgment set under the (hypothetical) agenda  $X \cup X'$ , the (feasible) agenda  $X$  resp. the (feasible) agenda  $X'$ . While basic agenda-insensitivity requires independence and coherence, full agenda-insensitivity requires stronger versions of independence and coherence. How are these stronger versions defined? First, an aggregation rule  $F$  for an agenda  $X$  is called **independent on  $Y$**  ( $\subseteq \bar{X}$ ) if the collective judgment on any proposition in  $Y$  only depends on the individuals' judgments on *this* proposition: for all propositions  $p \in Y$  and all profiles  $(J_1, \dots, J_n)$  and  $(J'_1, \dots, J'_n)$  in the domain, if for each individual  $i$   $J_i$  entails  $p$  if and only if  $J'_i$  entails  $p$ , then  $F(J_1, \dots, J_n)$  entails  $p$  if and only if  $F(J'_1, \dots, J'_n)$  entails  $p$  (see Dietrich 2006). Setting  $Y = X$  yields standard independence. Full agenda-insensitivity however requires independence on the scope  $Y = \bar{X}$ ; this is the 'maximal' choice of  $Y$ .

Second, I strengthen the coherence condition by requiring the aggregation rule to change coherently not just when the agenda setter extends the agenda, but more generally when he 'essentially extends' the agenda, i.e., when he extends the scope of the agenda:

**Definition 13** *An aggregation system  $(F_X)_{X \in \mathcal{X}}$  is **strongly coherent** if whenever an agenda  $X \in \mathcal{X}$  is essentially extended to another  $X' \in \mathcal{X}$ , i.e.,  $X \subseteq \bar{X}'$ , then the rule  $F_{X'}$  coheres with  $F_X$ .*

Note that if  $X \subseteq \bar{X}'$  (or equivalently,  $\bar{X} \subseteq \bar{X}'$ ), then the judgments for  $X'$  subsume those for  $X$ : each  $J' \in \mathcal{J}_{X'}$  entails a  $J \in \mathcal{J}_X$ . This implies a concrete characterization of coherence of  $F_{X'}$  with  $F_X$  in case  $X'$  essentially extends  $X$ :

**Remark 2** *In case  $X'$  essentially extends  $X$ , i.e.,  $X \subseteq \bar{X}'$ , coherence of  $F_{X'}$  with  $F_X$  means that the outcome of  $F_{X'}$  entails that of  $F_X$  for at least some 'essential extensions' of the individual judgments: for any  $J_1, \dots, J_n \in \mathcal{J}_X$ ,  $F_{X'}(J'_1, \dots, J'_n)$  entails  $F_X(J_1, \dots, J_n)$  for at least some  $J'_1, \dots, J'_n \in \mathcal{J}_{X'}$  entailing  $J_1, \dots, J_n$ , respectively.<sup>14</sup>*

I can now state the characterization of full agenda-insensitivity:

**Theorem 3** *An aggregation system  $(F_X)_{X \in \mathcal{X}}$  is fully agenda-insensitive if and only if it is independent on the entire scope  $\bar{X}$  and strongly coherent.*

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<sup>14</sup>Strong coherence is equivalent to ordinary coherence if the scope of any feasible agenda is a feasible agenda, i.e., if  $X \in \mathcal{X} \Rightarrow \bar{X} \in \mathcal{X}$ .

One may regard Theorems 2 and 3 as formal counterparts of claims in Dietrich (2006) about the role of independence and independence on the scope in preventing agenda manipulation, although Dietrich (2006) does not yet invoke feasible agendas, aggregation systems, and coherence or strong coherence.

Strong coherence has an interesting consequence. If two agendas  $X$  and  $X'$  are equivalent (i.e., have same scope  $\overline{X} = \overline{X'}$ ), any judgment set for  $X$  is equivalent to one for  $X'$ , and any aggregation rule for  $X$  is equivalent to one for  $X'$ . Formally, any  $J \in \mathcal{J}_X$  is **equivalent** to the unique  $J^* \in \mathcal{J}_{X'}$  such that  $J$  and  $J^*$  entail each other; and any aggregation rule  $F : \mathcal{J}_X^n \rightarrow \mathcal{J}_X$  is **equivalent** to the unique rule  $F' : \mathcal{J}_{X'}^n \rightarrow \mathcal{J}_{X'}$  defined as the image of  $F$  via transforming judgment sets in  $\mathcal{J}_X$  into equivalent ones in  $\mathcal{J}_{X'}$ ; formally,

$$[F(J_1, \dots, J_n)]^* = F'(J_1^*, \dots, J_n^*) \text{ for all } J_1, \dots, J_n \in \mathcal{J}_X.$$

One easily checks that strong coherence ensures equal treatment of equivalent agendas (and thus prevents Dietrich's 2006 'logical agenda manipulation'). Formally:

**Remark 3** *If an aggregation system  $(F_X)_{X \in \mathcal{X}}$  is strongly coherent, then for any equivalent agendas  $X, X' \in \mathcal{X}$  the corresponding rules  $F_X$  and  $F_{X'}$  are equivalent.*

I now consider a third agenda-insensitivity condition. Rather than requiring that *all* collective judgments in the scope are irreversible (by a change of agenda), let us merely require irreversibility of those collective judgments which are *most important* or *focal* in the sense of being unanimously supported by all individuals. Indeed, reversing a unanimously supported collective judgment seems particularly bad, as it goes against ('overrules') *all* individuals. The condition that unanimously supported collective judgments cannot be reversed by agenda manipulation is formally stated as follows:

**Definition 14** *An aggregation system  $(F_X)_{X \in \mathcal{X}}$  is **focally agenda-insensitive** if any two feasible agendas  $X, X' \in \mathcal{X}$  lead to the same collective judgment on any unanimously accepted proposition in  $\overline{X} \cap \overline{X'}$ : for all  $J_1, \dots, J_n \in \mathcal{J}_{X \cup X'}$  and all propositions  $p \in \overline{X} \cap \overline{X'}$  entailed by each  $J_i$ ,<sup>15</sup>*

$$F_X(J_1 \cap X, \dots, J_n \cap X) \text{ entails } p \Leftrightarrow F_{X'}(J_1 \cap X', \dots, J_n \cap X') \text{ entails } p.$$

This condition turns out to be equivalent to the requirement that each rule  $F_X$  is implicit consensus preserving, assuming a mild condition of non-degeneracy (i.e., unanimity preservation):

**Theorem 4** *An aggregation system  $(F_X)_{X \in \mathcal{X}}$  is focally agenda-insensitive and unanimity preserving if and only if it is implicit consensus preserving.*

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<sup>15</sup>Note that  $J_i$  entails  $p$  if and only if  $J_i \cap X$  entails  $p$  (since  $p \in \overline{X}$ ), and if and only if  $J_i \cap X'$  entails  $p$  (since  $p \in \overline{X'}$ ). So, the requirement that each  $J_i$  entails  $p$  means that  $p$  emerges as unanimously accepted, whether agenda  $X$  or agenda  $X'$  is used.

## 5 Agenda-insensitive aggregation: impossibility results

Our characterization results (Theorems 2-4) establish that agenda-insensitivity requires that aggregation rules satisfy certain axioms such as independence. But such axioms imply dictatorial aggregation for many agendas, by Theorem 1 and two other theorems of the literature. This turns our characterization results into impossibility results about agenda-insensitive aggregation. These impossibility results are now stated as corollaries.

I begin with basic agenda-insensitivity. By Theorem 2, this condition implies independence. However, by the well-known Arrow-like theorem in the field, independent aggregation rules must, for certain agendas, be degenerate, i.e., either dictatorial or not unanimity preserving. Formally:

**Theorem 5** (*Dietrich-List 2007a, Dokow-Holzman 2010<sup>16</sup>*) *Given an agenda, all independent and unanimity preserving aggregation rules  $F : \mathcal{J}^n \rightarrow \mathcal{J}$  are dictatorial if and only if the agenda is ‘strongly connected’.*

Which agendas count as ‘strongly connected’? The most important conceptual point is that the class of these agendas is far smaller than (and included in) the class of ‘impossibility agendas’ in Theorem 1, i.e., the class of non-nested non-tiny agendas. For instance, an agenda of type  $X = \{p, q, p \cap q\}^\pm$  is *not* strongly connected (so escapes the Arrow-like impossibility), though it is non-nested and non-tiny (so falls into Theorem 1’s impossibility). Formally, strong connectedness is the conjunction of two well-known conditions, *pathconnectedness* (introduced by Nehring and Puppe 2002 under the label ‘total blockedness’) and *pair-negatability*, which are in turn defined as follows:

- *Pathconnectedness*: Recall first that a proposition  $p \in X$  *conditionally entails* another  $q \in X$  – written  $p \vdash^* q$  – if  $\{p\} \cup Y$  entails  $q$  for some set  $Y \subseteq X$  which (for non-triviality) is consistent with  $p$  and with  $\bar{q}$ . Agenda  $X$  is *pathconnected* if for any propositions  $p, q \in X \setminus \{\Omega, \emptyset\}$  there are  $p_1, \dots, p_m \in X$  ( $m \geq 1$ ) such that  $p = p_1 \vdash^* p_2 \vdash^* \dots \vdash^* p_m = q$ . Some important agendas are pathconnected, but many others are not.<sup>17</sup>
- *Pair-negatability*: Recall further that a set  $Y \subseteq X$  is *minimal inconsistent* if it is inconsistent but each proper subset of  $Y$  is consistent. The agenda  $X$  is *pair-negatable* if it has a minimal inconsistent subset  $Y \subseteq X$  which becomes consistent after negating some two members (i.e.,  $(Y \setminus \{p, q\}) \cup \{\bar{p}, \bar{q}\}$  is consistent for some distinct  $p, q \in Y$ ). Most concrete agendas are pair-negatable.<sup>18</sup>

Prominently, the preference agenda is strongly connected, and is thus subject to the

<sup>16</sup>Both papers build on Nehring and Puppe (2002/2010). Dietrich and List prove only the ‘if’ direction, and Dokow and Holzman prove both directions.

<sup>17</sup>The preference agenda for more than two alternatives is pathconnected: e.g.,  $xPy$  entails  $xPz$  conditionally on  $Y = \{yPz\}$  (for distinct options  $x, y, z$ ). But an agenda of the form  $X = \{p, q, p \cap q\}^\pm$  is not pathconnected: check that none of  $\bar{p}, \bar{q}, \overline{p \cap q}$  conditionally entails any of  $p, q, p \cap q$  (assuming  $\emptyset, \Omega \notin X$ ). Also the agendas in (1) are not pathconnected.

<sup>18</sup>The preference agenda for more than two alternatives is pair-negatable:  $Y = \{xPy, yPz, zPx\} \subseteq X$  is minimal inconsistent and becomes consistent after replacing  $xPy$  by  $yPx$  and  $yPz$  by  $zYy$ . Also  $X = \{p, q, p \cap q\}^\pm$  is pair-negatable (assuming  $\emptyset, \Omega \notin X$ ):  $Y = \{p, q, \overline{p \cap q}\}$  is minimal inconsistent but  $\{\bar{p}, \bar{q}, \overline{p \cap q}\}$  is consistent, and also  $Y = \{\bar{p}, p \cap q\}$  is minimal inconsistent but  $\{p, \overline{p \cap q}\}$  is consistent. (In the definition of pair-negatability, one can equivalently replace ‘some two members’ by ‘some positive even number of members’.)

impossibility, as is already known from Arrow's Theorem (to which Theorem 5 indeed reduces in the case of the preference agenda).

Given this Arrow-like theorem, our Theorem 2 immediately implies that agenda-insensitive aggregation systems must be degenerate:

**Corollary 1** *If an aggregation system  $(F_X)_{X \in \mathcal{X}}$  is agenda-insensitive and unanimity preserving, then the rule  $F_X$  is dictatorial for each strongly connected agenda  $X \in \mathcal{X}$  (more generally, each agenda  $X \in \mathcal{X}$  included in a strongly connected agenda  $X' \in \mathcal{X}$ ).<sup>19</sup>*

Let us now turn to the condition of *full* agenda-sensitivity, which by Theorem 3 forces to independence on the scope. Unfortunately, no non-degenerate aggregation rules are independent on the scope, as long as the agenda is **non-monadic** i.e., contains more than one issue  $\{p, \bar{p}\}$  ( $\neq \{\Omega, \emptyset\}$ ):

**Theorem 6** (Dietrich 2006, Corollary 1) *Given an agenda, all aggregation rules  $F : \mathcal{J}^n \rightarrow \mathcal{J}$  which are independent on the scope are dictatorial or constant if and only if the agenda is non-monadic.<sup>20</sup>*

Given this result, Theorem 3 implies that fully agenda-insensitive aggregation systems must be degenerate:

**Corollary 2** *If an aggregation system  $(F_X)_{X \in \mathcal{X}}$  is fully agenda-insensitive, then the rule  $F_X$  is dictatorial or constant for each non-monadic agenda  $X \in \mathcal{X}$  (more generally, each agenda  $X \in \mathcal{X}$  included in the scope of a non-monadic agenda  $X' \in \mathcal{X}$ ).*

Finally, we turn to focal agenda-insensitivity, which by Theorem 4 forces to implicit consensus preserving (under the mild assumption of unanimity preservation). This by itself does not lead into an impossibility result. But if one combines focal with basic agenda-insensitivity, then one is forced to independence (by Theorem 2) as well as implicit consensus preserving (by Theorem 4), which leads us straight into the impossibility of Theorem 1. Formally:

**Corollary 3** *If an aggregation system  $(F_X)_{X \in \mathcal{X}}$  is basically and focally agenda-insensitive, and unanimity preserving, then the rule  $F_X$  is dictatorial for each non-nested non-tiny agenda  $X \in \mathcal{X}$  (more generally, each agenda  $X \in \mathcal{X}$  included in a non-nested non-tiny agenda  $X' \in \mathcal{X}$ ).*

## 6 Conclusion

I begin by summing up. I have firstly derived a new impossibility theorem on judgment aggregation, based on the familiar independence axiom and a particularly strong consensus axiom. Subsequently, I have defined and axiomatically characterized three

<sup>19</sup>The generalization mentioned in brackets holds because  $F_X$  inherits dictatorship or constancy from  $F_{X'}$  by coherence. Analogous remarks apply to Corollary 2 (in which the generalization draws on strong coherence) and Corollary 3.

<sup>20</sup>Dietrich (2006) only states the 'if' direction; but the 'only if' direction holds trivially.

types of agenda-insensitive aggregation: basic, full and focal agenda-insensitivity. Finally, combining these characterization results with the impossibility result (and two well-known impossibility results), I have derived impossibility results about each type of agenda-insensitivity.

Let me finish by mentioning another type agenda-insensitivity, which pertains not to particular procedures (aggregation systems), but to axioms on aggregation rules. I call an axiom **description-insensitive** if whenever two agendas  $X$  and  $X'$  are equivalent (i.e., have the same scope), then an aggregation rule for agenda  $X$  satisfies the axiom if and only if the equivalent rule for agenda  $X'$  (defined in Section 4) satisfies the axiom.<sup>21</sup> One might favour description-insensitive axioms on the grounds that any dependence on how the decision problem is framed is a form of arbitrariness. The standard unanimity principle is not description-invariant: respecting unanimity on propositions in  $X = \{p, q\}^\pm$  is considerably different from doing so for the equivalent agenda  $X = \{p \cap q, \bar{p} \cap q, p \cap \bar{q}, \bar{p} \cap \bar{q}\}^\pm$ . Our stronger consensus axiom – implicit consensus preservation – avoids this flaw; it is from *this* perspective more canonical. The following table classifies axioms according to whether they are description-invariant.

axiom	description-invariant?
unanimity principle	no
unanimity preservation	yes
implicit consensus preservation	yes
independence	no
independence on the scope	yes
anonymity	yes

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<sup>21</sup>To be entirely precise, one can identify an axiom with the set of all aggregation rules satisfying it, i.e., the set  $\mathcal{A} = \{F : \text{for some agenda } X \subseteq 2^\Omega, F \text{ is a rule } \mathcal{J}_X^\Omega \rightarrow \mathcal{J}_X \text{ satisfying the axiom}\}$ . An axiom is thus given by a set  $\mathcal{A}$  of rules  $F : \mathcal{J}_X^\Omega \rightarrow \mathcal{J}_X$  for agendas  $X \subseteq 2^\Omega$ . The axiom is description-insensitive if the set  $\mathcal{A}$  is closed under equivalence of rules.



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## A A more standard judgment-aggregation framework for the results and concepts of the paper

All displayed results of the main text (the ‘theorems’, ‘propositions’, ‘corollaries’ and ‘remarks’) and all definitions of properties of aggregation rules/systems (such as ‘independence’ and ‘basic/full/focal agenda-insensitivity’) continue to apply *as stated* under a more standard, non-semantic judgment-aggregation framework. Section A.1 defines the usual notions of this framework (following List and Pettit 2002 and more precisely Dietrich 2007/2014). Sections A.2 and A.3 add the notions of ‘scope’ and ‘aggregation system’, whose definitions are less obvious than in a semantic or syntactic framework.

### A.1 The common concepts

We still consider a group of individuals  $i = 1, \dots, n$  with  $n \geq 3$ . No underlying set of worlds  $\Omega$  is introduced. Instead, I define agendas from scratch:

**Definition 15** An **agenda** is a non-empty set  $X$  (of ‘propositions’) which is endowed with the notions of negation and interconnections, i.e.,

- (a) to each  $p \in X$  corresponds a proposition denoted  $\neg p \in X$  (‘not  $p$ ’) with  $\neg p \neq p = \neg\neg p$  (so  $X$  is partitioned into pairs  $\{p, \neg p\}$ , called ‘issues’),
  - (b) certain judgment sets  $J \subseteq X$  containing a single member of each issue count as ‘rational’, the non-empty set of them being denoted  $\mathcal{J}$  or  $\mathcal{J}_X$ ,
- where (in this paper)  $X$  is finite and  $|\mathcal{J}| > 1$ .<sup>22</sup>

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<sup>22</sup>Most results do not require the finiteness restriction. The condition that  $|\mathcal{J}| > 1$  excludes trivial agendas.

Notationally, an agenda will be denoted simply by its set of propositions ‘ $X$ ’, suppressing the structure on  $X$ .<sup>23</sup> The semantic agendas of the main text are a special case: there propositions are sets of worlds, with the set-theoretic notions of negation and interconnections.<sup>24</sup> Syntactic agendas are another special case: here propositions are sentences of a formal logic, with the logical notions of negation and interconnections.<sup>25</sup> Examples of syntactic agendas are  $X = \{a, b, a \wedge b\}^\pm$  and  $X = \{a, a \rightarrow b, b\}^\pm$ . Here and in what follows,  $Y^\pm$  still stands for  $\bigcup_{p \in Y} \{p, \neg p\}$ .

A judgment set  $J \subseteq X$  is **complete** if it contains a member of each issue  $\{p, \neg p\}$ , and **consistent** if it is extendable to a rational judgment set. So the *complete and consistent* judgment sets are precisely the *rational* judgment sets. In most concrete agendas, every proposition  $p \in X$  is **contingent**, i.e., neither a **contradiction** (for which  $\{p\}$  is inconsistent), nor a **tautology** (for which  $\{\neg p\}$  is inconsistent). A proposition  $p \in X$  (or set  $S \subseteq X$ ) **entails** another proposition  $p' \in X$  (or set  $S' \subseteq X$ ) if every rational judgment set containing  $p$  (resp. including  $S$ ) also contains  $p'$  (resp. includes  $S'$ ).

**Aggregation rules** for an agenda  $X$  are still functions  $F$  mapping any profile of ‘individual’ judgment sets  $(J_1, \dots, J_n)$  (from some domain, usually  $\mathcal{J}^n$ ) to a ‘collective’ judgment set  $F(J_1, \dots, J_n)$ .

The results of the main text draw on some agenda properties, whose general definitions are easily stated:

- A agenda  $X$  is **nested** if it takes the form  $X = \{p_1, \dots, p_m\}^\pm$ , where  $m$  is the number of issues and  $p_1$  entails  $p_2$ ,  $p_2$  entails  $p_3$ , and so on.
- An agenda  $X$  is **non-tiny** if it has *more than two* issue, i.e., more than four propositions (not counting non-contingent proposition if any, and identifying any equivalent propositions if any<sup>26</sup>).
- An agenda is **non-monadic** if it has *more than one* issue, i.e., more than two propositions (again not counting any non-contingent proposition and identifying any equivalent propositions).
- **Strong connectedness** (the conjunction of pathconnectedness and pair-negatability) is defined as before, modulo replacing complements  $\bar{p}$  by negations  $\neg p$  and replacing  $X \setminus \{\Omega, \emptyset\}$  by  $\{p \in X : p \text{ is contingent}\}$ .

<sup>23</sup>In more explicit algebraic terms, the agenda is the triple  $(X, \neg, \mathcal{J})$  containing  $X$  and the structure on  $X$ , i.e., the negation operator  $\neg$  and the set of rational judgment sets  $\mathcal{J}$ . Since the negation operator  $\neg$  (a mapping  $p \mapsto \neg p$  satisfying  $\neg p \neq p = \neg \neg p$ ) and the set of issues (a partition of  $X$  into binary sets) are interdefinable objects, we could equivalently define an agenda as a set  $X$  endowed with ‘issues and interconnections’, and define the negation of  $p$  as the unique proposition  $\neg p$  such that  $\{p, \neg p\}$  is an issue. Algebraically, the agenda would then be the structure  $X \equiv (X, \mathcal{I}, \mathcal{J})$ , where  $\mathcal{I}$  is the set of issues.

<sup>24</sup>I.e.,  $\neg p$  is the complement  $\bar{p}$ , and  $\mathcal{J}$  consists of those sets  $J \subseteq X$  (with a single member of each issue) for which  $\bigcap_{p \in J} p \neq \emptyset$ .

<sup>25</sup>Negation is given by the negation symbol (in fact, to ensure that double-negations cancel out, ‘ $\neg p$ ’ is defined from  $p$  by adding *or deleting* an initial negation symbol, depending on whether  $p$  already starts with a negation symbol).  $\mathcal{J}$  contains the logically consistent sets  $J \subseteq X$  containing a single member of each issue.

<sup>26</sup>In practice, agendas of course contain no non-contingent propositions and no two equivalent propositions. Propositions are equivalent if they entail one another.

## A.2 Closed agendas and scope of agendas

Extending agendas is easy in the special case of semantic agendas (it suffices to consider new subsets of  $\Omega$ ) or syntactic agendas (it suffices to consider new sentences of the logic). But how does this work in our general framework? In principle, there are two – as we shall see, equivalent – strategies for defining the scope of an agenda  $X$ . They can be stated informally as follows:

- (a) add to  $X$  all propositions (in an extended sense) which are constructible by combining propositions in  $X$ ;
- (b) add to  $X$  all propositions (in an extended sense) on which the judgment (‘yes’ or ‘no’) is determined by the judgments on the propositions in  $X$ .

I choose the approach (a), but also briefly discuss the approach (b) (which is perhaps more in line with Dietrich’s 2006 original definition). Starting with some natural terminology, I call an agenda

- **closed under conjunction** if for any propositions  $p, q \in X$  there exists a proposition  $r$  (the **conjunction** of  $p$  and  $q$ ) such that any rational judgment set contains  $r$  if and only if it contains both  $p$  and  $q$ ,
- **closed under disjunction** if for any propositions  $p, q \in X$  there exists a proposition  $r$  (the **disjunction** of  $p$  and  $q$ ) such that any rational judgment set contains  $r$  if and only if it contains  $p$  or  $q$  (possibly both).

These two closure properties are in fact equivalent, as is seen shortly. Most relevant agendas, including the semantic ones in the main text, are **redundancy-free**: they contain no two distinct propositions that are equivalent, i.e., entail each other.

**Proposition 2** *In any redundancy-free agenda  $X$ , the conjunction resp. disjunction of two propositions  $p$  and  $q$ , if existing, is unique and denoted by  $p \wedge q$  resp.  $p \vee q$ .*

**Proposition 3** *An agenda  $X$  is closed under conjunction if and only if it is closed under disjunction. I then call it **closed** simpliciter.*

To ‘close’ an agenda, we must extend it. First, let me spell out the obvious:

**Definition 16** *An agenda  $X$  is a **subagenda** of another  $X'$ , and  $X'$  an **extension** or **superagenda** of  $X$ , if  $X \subseteq X'$ , where the notions of negation and interconnections for  $X$  are those for  $X'$  restricted to  $X$  (i.e., the negation operator of  $X$  is that of  $X'$  restricted to  $X$ , and  $\mathcal{J}_X = \{J \cap X : J \in \mathcal{J}_{X'}\}$ ).*

The following result ensures that closing an agenda is possible in a unique way:

**Proposition 4** *Every agenda  $X$  has a closure, i.e., a minimal closed superagenda; it is moreover unique up to relabelling<sup>27</sup>.*

**Definition 17** *The **scope** of an agenda  $X$  is its (up to relabelling uniquely existing) minimal closed superagenda, denoted  $\overline{X}$ .*

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<sup>27</sup>Uniqueness up to relabelling means that between any two minimal closed superagendas  $X'$  and  $X''$  there exists an (agenda-)isomorphism that is constant on  $X$ , where an isomorphism is of course a bijection  $f : X' \rightarrow X''$  that preserves (i) the notion of negation (i.e.,  $f(\neg p) = \neg f(p)$  for all  $p \in X$ ) and (ii) the notion of interconnections (i.e.,  $J \in \mathcal{J}_X \Leftrightarrow f(J) \in \mathcal{J}_{X'}$  for all  $J \subseteq X$ ).

The judgments within an agenda  $X$  determine those within the entire scope  $\overline{X}$ . Formally:

**Definition 18** (a) Given an agenda, a judgment set  $J$  **settles** a proposition  $p$  if  $J$  entails  $p$  or entails  $\neg p$ .  
(b) An agenda  $X$  **settles** a superagenda  $X'$  if each rational judgment set  $J \in \mathcal{J}_X$  settles each proposition in  $X'$  (equivalently, is uniquely extendable to a  $J' \in \mathcal{J}_{X'}$ ).

**Proposition 5** Every agenda settles its scope.

In fact, a stronger result can be shown: the scope of an agenda  $X$  is the (up to relabelling unique) *maximal* superagenda which is settled by  $X$  (and is redundancy-free outside  $X$ , i.e., contains no two equivalent propositions outside  $X$ ). We could therefore have used an alternative and equivalent definition of the scope:

**Definition 19 (alternative statement)** The **scope** of an agenda  $X$  is the (up to relabelling uniquely existing) maximal superagenda settled by  $X$  and redundancy-free outside  $X$ .

The following lemma gives a concrete idea of the totality of propositions in the scope. It uses conjunctions/disjunctions of *any number* of propositions, which are defined like conjunctions/disjunctions of two propositions.<sup>28</sup>

**Lemma 1** Every proposition  $p$  in the scope of a redundancy-free agenda  $X$  is a disjunction of conjunctions of propositions in  $X$ ; for instance,  $p = \bigvee_{J \in \mathcal{J}^p} \bigwedge_{q \in J} q$ , where  $\mathcal{J}^p := \{J \in \mathcal{J}_X : J \text{ entails } p\}$ .

Finally, the scope carries a familiar algebraic structure:

**Proposition 6** Any closed redundancy-free agenda – for instance the scope of a redundancy-free agenda – is a Boolean algebra with respect to the relation of entailment between propositions, with the meet, join, and complement given by the conjunction, disjunction, and negation, respectively.<sup>29</sup>

Recall that Boolean algebras are defined as follows. First, a **lattice** is a partially ordered set  $\mathcal{L} \equiv (\mathcal{L}, \leq)$  such that any two elements  $p, q \in \mathcal{L}$  have a meet  $p \wedge q$  (greatest lower bound) and a join  $p \vee q$  (smallest upper bound). It is **distributive** if  $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$  and  $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$  for all  $p, q, r \in \mathcal{L}$ . A **Boolean algebra** is a distributive lattice  $(\mathcal{L}, \leq)$  such that  $\mathcal{L}$  contains a greatest element  $\top$  (the ‘top’ or ‘tautology’) and a bottom  $\perp$  (the ‘bottom’ or ‘contradiction’), and every element has an algebraic complement, i.e., an element whose join with  $p$  is  $\top$  and whose meet with  $p$  is  $\perp$ . The paradigmatic Boolean algebras are the set-theoretic ones: here there exists a set  $\Omega$

<sup>28</sup>Generalizing the earlier definition, I call a proposition  $r$  the **conjunction** (resp. **disjunction**) of a set of propositions  $S$  if any rational judgment set contains  $r$  if and only if it contains all (resp. some)  $p \in S$ .

<sup>29</sup>Without assuming redundancy-freeness, the agenda is a Boolean algebra modulo equivalence between propositions.

such that  $\mathcal{L} \subseteq 2^\Omega$ ,  $\leq = \subseteq$ ,  $\top = \Omega$ ,  $\perp = \emptyset$ , and the meet, join and complement are given by the set-theoretic intersection, union and complement. By Stone's representation theorem, every Boolean algebra is isomorphic to such a set-theoretic one. Another example is the Boolean algebra generated from a logic, i.e., the set of sentences modulo logical equivalence (where the logic includes classical negation and conjunction, which induce the algebraic negation, meet and join).

### A.3 Feasible agendas and aggregation systems

Just as in the main text, aggregation systems are families  $(F_X)_{X \in \mathcal{X}}$  containing an aggregation rule  $F_X : \mathcal{J}_X^n \rightarrow \mathcal{J}_X$  for each agenda  $X$  from a given set  $\mathcal{X}$  of 'feasible' agendas  $\mathcal{X}$  (see Definition 6). However, we need to say what 'feasible agendas' are in the present framework. In the semantic framework, they were subagendas of the 'universal' agenda  $2^\Omega$  which contains the totality of all propositions at the disposal of the agenda setter. Our present framework has no set of worlds  $\Omega$  and thus no pre-defined universal agenda. We therefore enrich the framework by assuming a ('universal') agenda  $\mathcal{L}$ , which we take to be closed and redundancy-free (so  $\mathcal{L}$  defines a Boolean algebra by Proposition 6). Think of  $\mathcal{L}$  as a reservoir of propositions. It could be as large as an entire language, or as small as a set of propositions on a relevant topic (such as a given court trial).

Now  $\mathcal{X}$  is simply a set of subagendas  $X$  of  $\mathcal{L}$ : those deemed **feasible/possible**. All we assume about  $\mathcal{X}$  is, here again, that it contains at least each binary agenda  $\{p, \neg p\} \subseteq \bigcup_{X \in \mathcal{X}} X$ . Since  $\mathcal{L}$  is closed, the scope  $\overline{X}$  of an agenda  $X \in \mathcal{X}$  is again a subagenda of  $\mathcal{L}$  (the smallest one that includes  $X$ ).<sup>30</sup> Note that quite possibly  $\overline{X} \notin \mathcal{X}$  since  $\overline{X}$  might be too rich and complex for being feasible.<sup>31</sup>

## B Proofs

All proofs are formulated for the general framework of Appendix A. The set of individuals is denoted  $N := \{1, \dots, n\}$ . Recall that for an agenda  $X$  the set of rational judgment sets is denoted ' $\mathcal{J}$ ' or sometimes, to avoid ambiguity, ' $\mathcal{J}_X$ '.

### B.1 Results of Section 3 on single aggregation rules

*Proof of Proposition 1.* For the agenda  $X$ , consider an aggregation rule  $F : \mathcal{J}^n \rightarrow \mathcal{J}$ . I write ICP1, ICP2 and ICP3 for the three versions of implicit consensus preservation, respectively.

'*ICP1*  $\Rightarrow$  *ICP3*': Assume ICP1. Consider any  $(J_1, \dots, J_n) \in \mathcal{J}^n$ . In the scope we can form the proposition  $p := \left( \bigwedge_{q \in J_1} q \right) \vee \dots \vee \left( \bigwedge_{q \in J_n} q \right)$  (i.e., the proposition that all  $q$  in  $J_1$  or all  $q$  in  $J_2$  ... or all  $q$  in  $J_n$  hold). Each  $J_i$  entails  $\bigwedge_{q \in J_i} q$ , and hence, entails

<sup>30</sup>Or so we may assume without loss of generality. Recall that the scope is unique up to relabelling propositions.

<sup>31</sup>All our results about aggregation systems remain true if we allow  $\mathcal{L}$  and any  $X \in \mathcal{X}$  to be infinite, i.e., to be agendas in a generalized sense without finiteness restriction.

$p$ . So,  $F(J_1, \dots, J_n)$  entails  $p$  by ICP1. Let  $J$  be the unique extension of  $F(J_1, \dots, J_n)$  to a set in  $\mathcal{J}_{\overline{X}}$ . Since  $F(J_1, \dots, J_n)$  entails  $p$ ,  $J$  contains  $p$ . So, for some  $i$ ,  $\bigwedge_{q \in J_i} q \in J$ , and

thus  $J_i \subseteq J$ . It follows that  $J_i = J \cap X = F(J_1, \dots, J_n)$ . QED

‘*ICP3*  $\Rightarrow$  *ICP2*’: Assume ICP3 and consider a feature  $\mathcal{K} \subseteq \mathcal{J}$  and a profile  $(J_1, \dots, J_n) \in \mathcal{J}^n$  such that  $J_1, \dots, J_n \in \mathcal{K}$ . By ICP3,  $F(J_1, \dots, J_n) \in \{J_1, \dots, J_n\}$ . So,  $F(J_1, \dots, J_n) \in \mathcal{K}$ . QED

‘*ICP2*  $\Rightarrow$  *ICP1*’: Assume ICP2. Consider any  $p \in \overline{X}$  and any profile  $(J_1, \dots, J_n) \in \mathcal{J}^n$  such that each  $J_i$  entails  $p$ . Since each  $J_i$  belongs to the feature  $\mathcal{K} := \{J \in \mathcal{J} : J \text{ entails } p\}$ , so does  $F(J_1, \dots, J_n)$  by ICP2. ■

As part of the proof of Theorem 1, I show several lemmas. For an agenda  $X$ , an aggregation rule  $F$  on  $\mathcal{J}^n$  is called **systematic** if there exists a set  $\mathcal{W}$  of (‘winning’) coalitions  $C \subseteq N$  such that

$$F(J_1, \dots, J_n) = \{p \in X : \{i : p \in J_i\} \in \mathcal{W}\} \text{ for all } J_1, \dots, J_n \in \mathcal{J}.$$

In this case, the set  $\mathcal{W}$  is uniquely determined and denoted by  $\mathcal{W}_F$ .

**Lemma 2** *Every independent and implicit consensus preserving aggregation rule  $F : \mathcal{J}^n \rightarrow \mathcal{J}$  is systematic if and only if the agenda  $X$  is non-nested.*

*Proof.* Let  $X$  be an agenda. We may assume without loss of generality that all  $p \in X$  are contingent, because each side of the claimed equivalence remains true (or false) if the non-contingent propositions are removed from the agenda.

1. In this part we assume that  $X$  is non-nested and consider an independent and implicit consensus preserving rule  $F : \mathcal{J}^n \rightarrow \mathcal{J}$ . I show that  $F$  is systematic (drawing on Dietrich and List 2013). For any  $p, q \in X$ , I define  $p \sim q$  to mean that there exists a finite sequence  $p_1, \dots, p_k \in X$  with  $p_1 = p$  and  $p_k = q$  such that any neighbours  $p_l, p_{l+1}$  are not exclusive (i.e.,  $\{p_l, p_{l+1}\}$  is consistent) and not exhaustive (i.e.,  $\{\neg p_l, \neg p_{l+1}\}$  is consistent). I prove five claims: the first four gradually establish that  $p \sim q$  for all  $p, q \in X$ , and the last shows that  $F$  is systematic.

*Claim 1:* For all  $p, q \in X$ ,  $p \sim q \Leftrightarrow \neg p \sim \neg q$ .

It suffices to show one direction of implication, as  $\neg \neg p = p$  for all  $p \in X$ . Let  $p, q \in X$  with  $p \sim q$ . Then there is a path  $p_1, \dots, p_k \in X$  between  $p$  to  $q$  where any neighbours  $p_j, p_{j+1}$  are non-exclusive and non-exhaustive. To see why  $\neg p \sim \neg q$ , note that  $\neg p_1, \dots, \neg p_k$  is a path between  $\neg p$  and  $\neg q$  where any neighbours  $\neg p_j, \neg p_{j+1}$  are non-exclusive (as  $p_j, p_{j+1}$  are non-exhaustive) and non-exhaustive (as  $p_j, p_{j+1}$  are non-exclusive). QED

*Claim 2:* If  $p \in X$  entails  $q \in X$ , then  $p \sim q$ .

If  $p \in X$  entails  $q \in X$ , then  $p \sim q$  in virtue of a direct connection:  $p, q$  are neither exclusive nor exhaustive (for instance,  $\{p, q\}$  is consistent because  $p$  is not a contradiction and entails  $q$ ). QED

*Claim 3:*  $\sim$  is an equivalence relation on  $X$ , and for all  $p, q \in X$ ,  $p \sim q$  or  $p \sim \neg q$ . (So each equivalence class contains at least one member of each issue  $\{q, \neg q\}$ , and it is the only equivalence class if it contains both members of some issue.)

Reflexivity, symmetry and transitivity are all obvious (where reflexivity uses that every  $p \in X$  is contingent). Now consider  $p, q \in X$  such that  $p \not\sim q$ ; we have to show that  $p \sim \neg q$ . Since  $p \not\sim q$ ,  $\{p, q\}$  or  $\{\neg p, \neg q\}$  is inconsistent. In either case, one of  $p$  and  $\neg q$  entails the other, so that  $p \sim \neg q$  by Claim 2. QED

*Claim 4:*  $p \sim q$  for all  $p, q \in X$ .

Let  $X_+$  be an equivalence class w.r.t.  $\sim$  and suppose for a contradiction that  $X_+ \neq X$ . Then, by Claim 3,  $X_+$  must contain exactly one member of each issue  $\{r, \neg r\}$ . We show that  $X_+$  is weakly ordered by the entailment relation between propositions – implying that  $X$  is nested, a contradiction. As the entailment relation on  $X_+$  is of course transitive, it remains to show that it is complete on  $X_+$ . So we consider  $p, q \in X_+$ , and have to show that  $p$  entails  $q$  or  $q$  entails  $p$ . We have  $p \not\sim \neg q$ , since otherwise  $X_+$  would include the entire issue  $\{q, \neg q\}$ . So  $\{p, \neg q\}$  or  $\{\neg p, q\}$  is inconsistent. Hence,  $p$  entails  $q$  or  $q$  entails  $p$ . QED

*Claim 5:*  $F$  is systematic.

Since  $F$  is independent, there exists a family  $(\mathcal{W}_p)_{p \in X}$  of sets of coalitions such that

$$F(J_1, \dots, J_n) = \{p \in X : \{i : p \in J_i\} \in \mathcal{W}_p\} \text{ for all } J_1, \dots, J_n \in \mathcal{J}. \quad (2)$$

It suffices to show that  $\mathcal{W}_p$  is the same for all  $p \in X$ . By Claim 4 and the definition of  $\sim$ , it suffices to show that  $\mathcal{W}_p = \mathcal{W}_q$  for all  $p, q \in X$  which are non-exclusive and non-exhaustive. Consider such  $p, q \in X$ . Consider any  $C \subseteq N$  and let us show that  $C \in \mathcal{W}_q \Leftrightarrow C \in \mathcal{W}_p$ . As  $\{p, q\}$  and  $\{\neg p, \neg q\}$  are consistent, there exist  $J_1, \dots, J_n \in \mathcal{J}$  such that  $p, q \in J_i$  for all  $i \in C$  and  $\neg p, \neg q \in J_i$  for all  $i \in N \setminus C$ . We now apply implicit consensus preservation, in any of its three variants. Using either variant 1 (and the fact that each  $J_i$  entails the proposition  $(p \wedge q) \vee (\neg p \wedge \neg q)$  in the scope), or variant 2 (and the fact that each  $J_i$  belongs to the feature  $\mathcal{K} := \{J \in \mathcal{J} : p \in J \Leftrightarrow q \in J\}$ ), or variant 3, it follows that  $p \in F(J_1, \dots, J_n) \Leftrightarrow q \in F(J_1, \dots, J_n)$ . By (2), the left side of this equivalence holds if and only if  $C \in \mathcal{W}_p$  and the right side holds if and only if  $C \in \mathcal{W}_q$ . So  $C \in \mathcal{W}_p \Leftrightarrow C \in \mathcal{W}_q$ . QED

2. Now assume that  $X$  is nested, i.e., of the form  $X = \{p_1, \dots, p_m\}^\pm$  where  $m$  is the number of issues and where  $p_1$  entails  $p_2$ ,  $p_2$  entails  $p_3$ , etc. I consider the aggregation rule  $F$  on  $\mathcal{J}^n$  defined as follows: for all  $J_1, \dots, J_n \in \mathcal{J}$ ,  $F(J_1, \dots, J_n)$  consists of each  $p_j$  contained in *all*  $J_i$  and each  $\neg p_j$  contained in *some*  $J_i$ . We have to show that  $F$  (i) maps into  $\mathcal{J}$ , (ii) is independent, (iii) is implicit consensus preserving, and (iv) is not systematic. The properties (ii) and (iv) are obvious (where (iv) uses that  $n > 1$  and that  $X$  contains a pair of contingent propositions  $p, \neg p$  because  $|\mathcal{J}| > 1$ ). It remains to prove (i) and (iii). Now (i) follows from (iii) by version 3 of implicit consensus preservation. To see why (iii) holds, note that for each  $J \in \mathcal{J}$  there is a cut-off level  $t \in \{1, \dots, m+1\}$  such that  $J = \{\neg p_1, \dots, \neg p_{t-1}, p_t, \dots, p_m\}$ , and that therefore for all  $J_1, \dots, J_n \in \mathcal{J}$  we have  $F(J_1, \dots, J_n) = J_i$  where  $i$  is the (or an) individual with highest cut-off level. ■

The next lemma is the main technical step towards Theorem 1 and provides two alternative characterizations of non-nested agendas. (Compare the characterization in (b) with the definition of *non-simple* agendas mentioned in Section 3: the only difference is that (b) allows  $Y$  to be consistent.)

**Lemma 3** *For any agenda  $X$ , the following are equivalent:*



- (a)  $X$  is non-nested (and non-tiny).
- (b)  $X$  has a subset  $Y$  such that  $|Y| \geq 3$  and  $(Y \setminus \{p\}) \cup \{\neg p\}$  is consistent for all  $p \in Y$ .
- (c)  $X$  has a subset  $Y$  such that  $|Y| = 3$  and  $(Y \setminus \{p\}) \cup \{\neg p\}$  is consistent for all  $p \in Y$ .

*Proof.* Let  $X$  be an agenda. I write  $p \vdash q$  to mean that  $p (\in X)$  entails  $q (\in X)$ , and  $S \vdash q$  to mean that  $S (\subseteq X)$  entails  $q$ . We may assume without loss of generality that  $X$  contains only contingent propositions, and is redundancy-free, i.e., contains no two equivalent propositions. The reason is that otherwise it suffices to do the proof for any redundancy-free subagenda containing only contingent propositions, because each of the conditions (a), (b) and (c) holds for  $X$  if and only if it holds for that subagenda; to see for instance why (b) holds for  $X$  if and only if it holds for the subagenda, note that  $(Y \setminus \{p\}) \cup \{\neg p\}$  can only be consistent for all  $p \in Y$  if  $Y$  contains no two equivalent propositions and no non-contingent propositions.

The equivalence between (b) and (c) is straightforward (to see why (b) implies (c), simply replace the set  $Y$  in (b) by a three-member subset of it). It is also relatively easy to see why (c) implies (a). Indeed, whenever (a) is violated, so is (c), by the following argument. First, if  $X$  is tiny, then (c) is violated since every three-element set  $Y \subseteq X$  takes the form  $Y = \{q, \neg q, p\}$  for some  $p, q \in X$ , and thus  $Y \setminus \{p\} \cup \{\neg p\}$  fails to be consistent. Second, if  $X$  is nested, say  $X = \{r, \neg r : r \in Z\}$  for some subset  $Z \subseteq X$  linearly ordered by entailment, condition (c) is violated since any three-element set  $Y \subseteq X$  has elements  $p \neq q$  which both belong to  $Z$  or both belong to  $\{\neg p : p \in Z\}$ , so that (by the linear orderedness of  $Z$  and of  $\{\neg r : r \in Z\}$  w.r.t. entailment)  $p \vdash q$  or  $q \vdash p$ , which implies that  $(Y \setminus \{q\}) \cup \{\neg q\}$  or  $(Y \setminus \{p\}) \cup \{\neg p\}$  is inconsistent.

It remains to show that (a) implies (c). Let  $X$  be non-nested and non-tiny; we show (c). We distinguish between two cases.

*Case 1:* no  $p, q \in X$  are logically independent, i.e., for no  $p, q \in X$  each of the sets  $\{p, q\}$ ,  $\{p, \neg q\}$ ,  $\{\neg p, q\}$  and  $\{\neg p, \neg q\}$  is consistent.

*Claim 1.1.* There is a (with respect to set-inclusion) maximal nested (sub)agenda  $X^* \subseteq X$ .

This follows from the fact that the set of nested subagendas  $V \subseteq X$  is non-empty (because it contains any single-issue subagenda  $\{p, \neg p\}$ ) and finite (because  $X$  is finite). QED

Since  $X^*$  is nested, we may write it as  $X^* = \bigcup_{p \in X_+^*} \{p, \neg p\}$  where  $X_+^*$  is a subset of  $X^*$  which contains exactly one member of each issue  $\{p, \neg p\} \subseteq X^*$  and is linearly ordered w.r.t. entailment.

*Claim 1.2.* There exists an  $s \in X \setminus X^*$  such that  $\{s, p\}$  is consistent for all  $p \in X_+^*$ .

Since  $X^*$  is nested but  $X$  is not, we have  $X^* \subsetneq X$ , and thus there are  $r, \neg r \in X \setminus X^*$ . It suffices to show that at least one of  $r$  and  $\neg r$  is consistent with each  $p \in X_+^*$ . This is true because otherwise there would exist  $p, p' \in X_+^*$  such that  $\{r, p\}$  and  $\{\neg r, p'\}$  are inconsistent, which (recalling that  $p \vdash p'$  or  $p' \vdash p$ , and writing  $p''$  for the logically stronger one of  $p$  and  $p'$ ) implies that  $\{r, p''\}$  and  $\{\neg r, p''\}$  are inconsistent, so that  $\{p''\}$  is inconsistent, a contradiction since  $p''$  is contingent. QED

I define

$$\begin{aligned} Y_1 & : = \{p \in X_+^* : p \vdash s\}, \\ Y_2 & : = \{p \in X_+^* : \neg p \vdash s\}. \end{aligned}$$

*Claim 1.3.*  $Y_1 \cap Y_2 = \emptyset$ , and  $Y_1 \cup Y_2 = X_+^*$ .

First,  $Y_1 \cap Y_2 = \emptyset$ , because otherwise there would be a  $p \in X_+^*$  such that  $p \vdash s$  and  $\neg p \vdash s$ , a contradiction as  $s$  is not a tautology. Second, suppose for a contradiction that  $p \in X_+^* \setminus (Y_1 \cup Y_2)$ . I ultimately show that the agenda  $X^* \cup \{s, \neg s\}$  is nested, a contradiction as  $X^*$  is a *maximal* nested subagenda of  $X$ .

Since  $p$  and  $s$  are not logically independent (by assumption of Case 1), and since  $\{p, s\}$  is consistent (by Claim 1.2),  $\{p, \neg s\}$  is consistent (as  $p \notin Y_1$ ) and  $\{\neg p, \neg s\}$  is consistent (as  $p \notin Y_2$ ), it follows that  $\{\neg p, s\}$  is inconsistent, so that  $s \vdash p$ . We next show that  $s$  entails not just in  $p$ , but also all other propositions in  $X_+^* \setminus Y_1$ :

$$s \vdash p' \text{ for all } p' \in X_+^* \setminus Y_1. \quad (3)$$

To show this, let  $p' \in X_+^* \setminus Y_1$ , and note first that  $\neg p'$  and  $\neg s$  are entailed by  $\{\neg p', \neg p, \neg s\}$ . Hence (as  $s \vdash p$ , i.e.,  $\neg p \vdash \neg s$ ),  $\neg p'$  and  $\neg s$  are entailed by  $\{\neg p', \neg p\}$ . So, since the set  $\{\neg p', \neg p\}$  is consistent (as either  $\neg p' \vdash \neg p$  or  $\neg p \vdash \neg p'$ ), the set  $\{\neg p', \neg s\}$  is also consistent. Since  $p'$  and  $s$  are not logically independent (by assumption of Case 1), and since  $\{p', s\}$  is consistent (by Claim 1.2),  $\{p', \neg s\}$  is consistent (as  $p' \notin Y_1$ ) and  $\{\neg p', \neg s\}$  is consistent (as just shown), it follows that  $\{\neg p', s\}$  is inconsistent, so that  $s \vdash p'$ . This proves (3).

Note that for every  $p'$  in  $X_+^*$ , either  $p' \vdash s$  (if  $p' \in Y_1$ ) or  $s \vdash p'$  (if  $p' \notin Y_1$ , by (3)). So the augmented (sub-)agenda  $X^* \cup \{s, \neg s\}$  is nested, a contradiction as  $X^*$  is a maximal nested subagenda of  $X$ . QED

*Claim 1.4.*  $Y_1, Y_2 \neq \emptyset$ .

By Claim 1.3 we may equivalently show that  $Y_1, Y_2 \neq X_+^*$ . Suppose for a contradiction that  $Y_1 = X_+^*$  or  $Y_2 = X_+^*$ . Then  $X^* \cup \{s, \neg s\}$  is a nested agenda, a contradiction since  $X^*$  was defined as a maximal nested subagenda of  $X$ . QED

The proof of condition (c) is completed by combining Claim 1.4 with the following observation:

*Claim 1.5.* For all  $q \in Y_1$  and  $r \in Y_2$ , the set  $Y := \{\neg q, r, s\}$  satisfies the requirements of condition (c), i.e.,  $|Y| = 3$  and  $(Y \setminus \{p\}) \cup \{\neg p\}$  is consistent for each  $p \in Y$ .

Consider any  $q \in Y_1$  and  $r \in Y_2$  and let  $Y := \{\neg q, r, s\}$ . To see why  $|Y| = 3$ , note that  $\neg q \neq r$  since  $r \in X_+^*$  while  $\neg q \notin X_+^*$ , and that  $s \neq \neg q, r$  since  $\neg q, r \in X^*$  while  $s \notin X^*$ . Further:

- $\{q, r, s\}$  is consistent, because, firstly,  $\{q, s\}$  is consistent by Claim 1.2, and, secondly,  $q \vdash r$ , as  $q$  and  $r$  belong to the linearly ordered set  $X_+^*$  and as  $r \not\vdash q$  (by the fact that  $q \in Y_1$  and  $r \notin Y_1$ ).
- $\{\neg q, \neg r, s\}$  is consistent, because, firstly,  $\neg r \vdash \neg q$  (since  $q \vdash r$ , as just shown), and, secondly,  $\neg r \vdash s$  (since  $r \in Y_2$ ).
- $\{\neg q, r, \neg s\}$  is consistent, because, firstly,  $\neg s \vdash \neg q$  (since  $q \vdash s$ , as  $q \in Y_1$ ), and, secondly,  $\neg s \vdash r$  (since  $\neg r \vdash s$ , as  $r \in Y_2$ ). QED

*Case 2:*  $p, q \in X$  are logically independent, i.e., all of  $\{p, q\}$ ,  $\{p, \neg q\}$ ,  $\{\neg p, q\}$  and  $\{\neg p, \neg q\}$  are consistent. Consider such  $p, q \in X$ . Since  $|X| > 4$  there is an  $r \in$

$X \setminus \{p, \neg p, q, \neg q\}$ . As  $r$  is non-contradictory, it can be consistently added to at least one of the (consistent) sets  $\{p, q\}$ ,  $\{p, \neg q\}$ ,  $\{\neg p, q\}$  and  $\{\neg p, \neg q\}$ . We may assume without loss of generality that  $\{p, \neg q, r\}$  is consistent (otherwise, simply interchange  $p$  with  $\neg p$  and/or  $q$  with  $\neg q$ ). The argument distinguishes between two subcases.

*Subcase 2.1:*  $\{\neg p, \neg q, \neg r\}$  and  $\{p, q, \neg r\}$  are both consistent. In this case, condition (c) holds for  $Y := \{p, \neg q, \neg r\}$ , since each of the sets  $\{\neg p, \neg q, \neg r\}$ ,  $\{p, q, \neg r\}$  and  $\{p, \neg q, r\}$  is consistent.

*Subcase 2.2:*  $\{\neg p, \neg q, \neg r\}$  or  $\{p, q, \neg r\}$  is inconsistent (perhaps both are). We assume without loss of generality that  $\{p, q, \neg r\}$  is inconsistent, i.e.,  $\{p, q\} \vdash r$ . (The proof is analogous in the other case.) There are three subsubcases.

*Subsubcase 2.2.1:*  $\{\neg p, q, \neg r\}$  and  $\{p, \neg q, \neg r\}$  are both consistent. Here, condition (c) holds for  $Y := \{p, q, \neg r\}$ , since each of the sets  $\{\neg p, q, \neg r\}$ ,  $\{p, \neg q, \neg r\}$  and  $\{p, q, r\}$  is consistent (the latter set being consistent because  $\{p, q\}$  is consistent and entails  $r$ ).

*Subsubcase 2.2.2:*  $\{p, q, \neg r\}$  is inconsistent. So  $\{\neg p, q\} \vdash r$ . As also  $\{p, q\} \vdash r$ , we have  $q \vdash r$ . We once again distinguish between cases:

- First assume  $\{\neg p, \neg q, \neg r\}$  is consistent. Then condition (c) holds with  $Y = \{\neg p, \neg q, r\}$ , because  $\{p, \neg q, r\}$ ,  $\{\neg p, q, r\}$  and  $\{\neg p, \neg q, \neg r\}$  are consistent (where  $\{\neg p, q, r\}$  is consistent as  $\{\neg p, q\}$  is consistent and  $q \vdash r$ ).
- Second assume  $\{\neg p, \neg q, \neg r\}$  is inconsistent, i.e.,  $\{\neg p, \neg q\} \vdash r$ . Since also  $q \vdash r$ , we have  $\neg r \vdash \neg q, p$ . Condition (c) holds with  $Y = \{p, \neg q, r\}$ , because  $\{\neg p, \neg q, r\}$  is consistent (as  $\{\neg p, \neg q\}$  is consistent and entails  $r$ ),  $\{p, q, r\}$  is consistent (as  $\{p, q\}$  is consistent and entails  $r$ ) and  $\{p, \neg q, \neg r\}$  is consistent (as  $\neg r \vdash \neg q, p$ ).

*Subsubcase 2.2.3:*  $\{p, \neg q, \neg r\}$  is inconsistent. (If in the following proof for the current subsubcase we interchange  $p$  and  $q$ , then we obtain an alternative, but longer, proof for Subsubcase 2.2.2.) Since  $\{p, \neg q, \neg r\}$  is inconsistent,  $\{p, \neg q\} \vdash r$ . As also  $\{p, q\} \vdash r$ , it follows that  $p \vdash r$ . We now show that

$$\begin{aligned} & (*) \quad \{\neg p, q, r\} \text{ and } \{\neg p, \neg q, \neg r\} \text{ are consistent} \\ \text{or} \quad & (**) \quad \{\neg p, \neg q, r\} \text{ and } \{\neg p, q, \neg r\} \text{ are consistent.} \end{aligned} \tag{4}$$

To show this, we assume that  $(*)$  is violated and show that  $(**)$  holds, by distinguishing between two cases:

- First, let  $\{\neg p, q, r\}$  be inconsistent. It follows, on the one hand, that  $\{\neg p, q, \neg r\}$  is consistent (as  $\{\neg p, q\}$  is consistent), and, on the other hand, that  $\{\neg p, \neg q, r\}$  is consistent (as otherwise, by the inconsistency of  $\{\neg p, q, r\}$ ,  $\{\neg p, r\}$  would be inconsistent, i.e.,  $r \vdash p$ , a contradiction since  $p \vdash r$  and  $p \neq r$ ). This proves  $(**)$ .
- Second, let  $\{\neg p, q, r\}$  be consistent. Then  $\{\neg p, \neg q, \neg r\}$  is inconsistent as  $(*)$  is violated. It follows, on the one hand, that  $\{\neg p, \neg q, r\}$  is consistent (as  $\{\neg p, \neg q\}$  is consistent), and, on the other hand, that  $\{\neg p, q, \neg r\}$  is consistent (as otherwise  $\{\neg p, \neg r\}$  would be inconsistent, i.e.,  $\neg r \vdash p$ , a contradiction since  $p \vdash r$ ). This proves  $(**)$ .

We can now prove condition (c). In the case of  $(*)$ , (c) holds with  $Y = \{\neg p, \neg q, r\}$ , since  $\{p, \neg q, r\}$  is consistent (as assumed without loss of generality under Case 2),  $\{\neg p, q, r\}$  is consistent (by  $(*)$ ) and  $\{\neg p, \neg q, \neg r\}$  is consistent (by  $(*)$ ). In the case of  $(**)$ , (c) holds with  $Y = \{\neg p, q, r\}$ , since  $\{p, q, r\}$  is consistent (as  $\{p, q\}$  is consistent and  $p \vdash r$ ),  $\{\neg p, \neg q, r\}$  is consistent (by  $(**)$ ) and  $\{\neg p, q, \neg r\}$  is consistent (by  $(**)$ ). ■

Drawing on Lemma 3, I next show that for *almost every* agenda the set of winning coalitions of a systematic and implicit consensus preserving aggregation rule defines an ultrafilter (which would not be true if implicit consensus preservation were replaced by the standard unanimity condition).

**Lemma 4** *Consider a systematic and implicit consensus preserving aggregation rule  $F : \mathcal{J}^n \rightarrow \mathcal{J}$  for an agenda  $X$ , and coalitions  $C, C' \subseteq N$ .*

- (a) *If  $X$  satisfies  $|\mathcal{J}| > 2$ , then  $[C \in \mathcal{W}_F \text{ and } C \subseteq C'] \Rightarrow C' \in \mathcal{W}_F$ .*
- (b) *If  $X$  is non-nested and non-tiny, then  $C, C' \in \mathcal{W}_F \Rightarrow C \cap C' \in \mathcal{W}_F$ .*
- (c)  *$C \in \mathcal{W}_F \Leftrightarrow N \setminus C \notin \mathcal{W}_F$ .*

Note that  $|\mathcal{J}| > 2$  if and only if  $X$  has more than two propositions (one issue),<sup>32</sup> a very mild assumption, satisfied notably by non-tiny agendas.

*Proof.* Let  $X, F, C$  and  $C'$  be as specified.

(a) Suppose  $|\mathcal{J}| > 2$ ,  $C \in \mathcal{W}_F$  and  $C \subseteq C'$ . We show that  $C' \in \mathcal{W}_F$ . As  $|\mathcal{J}| > 2$ , there exist contingent and pairwise non-equivalent propositions  $p, \neg p, q, \neg q \in X$ . There must exist a member of  $\{p, \neg p\}$  which entails neither  $q$  nor  $\neg q$ , as can be shown using that the propositions  $p, \neg p, q, \neg q$  are contingent and pairwise non-equivalent. Without loss of generality, we assume that  $p$  entails neither  $q$  nor  $\neg q$  (otherwise simply interchange  $p$  and  $\neg p$ ). So  $\{p, q\}$  and  $\{p, \neg q\}$  are each consistent. Note that at least one of  $\{\neg p, q\}$  and  $\{\neg p, \neg q\}$  is consistent, as  $\neg p$  is not a contradiction. Without loss of generality, we assume the latter (otherwise interchange  $q$  and  $\neg q$ ). To summarize, each of the sets  $\{p, q\}$ ,  $\{p, \neg q\}$  and  $\{\neg p, \neg q\}$  is consistent. We may therefore consider a profile  $(J_1, \dots, J_n) \in \mathcal{J}^n$  such that

$$J_i \supseteq \begin{cases} \{p, q\} & \text{for all } i \in C, \\ \{p, \neg q\} & \text{for all } i \in C' \setminus C, \\ \{\neg p, \neg q\} & \text{for all } i \in N \setminus C'. \end{cases}$$

First, since each  $J_i$  contains  $p$  or  $\neg q$ , so does  $F(J_1, \dots, J_n)$  by implicit consensus preservation (version 2). Second,  $q \in F(J_1, \dots, J_n)$  since  $\{i \in N : q \in J_i\} = C \in \mathcal{W}_F$ . These two facts imply that  $p \in F(J_1, \dots, J_n)$ . So, as  $\{i : p \in J_i\} = C'$ , we have  $C' \in \mathcal{W}_F$ .

(b) Suppose  $X$  is non-nested and non-tiny, and assume  $C, C^* \in \mathcal{W}_F$ . We show that  $C \cap C^* \in \mathcal{W}_F$ . By assumption on  $X$  and Lemma 3, there is a three-element set  $Y = \{p, q, r\} \subseteq X$  such that each of  $\{\neg p, q, r\}$ ,  $\{p, \neg q, r\}$  and  $\{p, q, \neg r\}$  is consistent. This allows us to construct a profile  $(J_1, \dots, J_n) \in \mathcal{J}^n$  such that

$$J_i \supseteq \begin{cases} \{\neg p, q, r\} & \text{if } i \in C \cap C^* \\ \{p, q, \neg r\} & \text{if } i \in C^* \setminus C \\ \{p, \neg q, r\} & \text{if } i \in N \setminus C^*. \end{cases}$$

First,  $q \in F(J_1, \dots, J_n)$  as  $\{i : q \in J_i\} = C^* \in \mathcal{W}_F$ . Second, as  $C \in \mathcal{W}_F$  and  $C \subseteq C \cup (N \setminus C^*)$ , we have  $C \cup (N \setminus C^*) \in \mathcal{W}_F$  by part (a); hence  $r \in F(J_1, \dots, J_n)$  as  $\{i : r \in J_i\} = C \cup (N \setminus C^*)$ . Third, as each  $J_i$  contains  $\neg p$  or  $\neg q$  or  $\neg r$ , so does  $F(J_1, \dots, J_n)$  by implicit consensus preservation (version 2). These three facts imply that  $\neg p \in F(J_1, \dots, J_n)$ . Hence, as  $\{i : \neg p \in J_i\} = C \cap C^*$ , we have  $C \cap C^* \in \mathcal{W}_F$ .

<sup>32</sup>counting only contingent propositions and counting equivalent propositions (if any) only once

(c) This claim is obvious, as (by  $|\mathcal{J}| > 1$ ) we can choose a contingent proposition  $p \in X$  and construct a profile in  $\mathcal{J}^n$  in which all  $i \in C$  accept  $p$  and all  $i \in N \setminus C$  accept  $\neg p$ . ■

I can now prove Theorem 1, whose ‘if’ part will follow from the above lemmas.

*Proof of Theorem 1.* 1. In this part of the proof, let the agenda  $X$  be non-nested and non-tiny, and let  $F : \mathcal{J}^n \rightarrow \mathcal{J}$  be independent and implicit consensus preserving. I need to show that  $F$  is dictatorial. By Lemma 2,  $F$  is systematic. By Lemma 4, the set of winning coalitions  $\mathcal{W}_F$  is an ultrafilter over the set of individuals  $N$ . As is well-known, every ultrafilter over a finite set is principal, i.e., there is an individual  $j \in N$  such that  $\mathcal{W}_F = \{C \subseteq N : j \in C\}$ . Clearly,  $j$  is a dictator.

2. Conversely, assume the agenda  $X$  is nested or tiny. I need to construct a non-dictatorial rule  $F : \mathcal{J}^n \rightarrow \mathcal{J}$  which is independent and implicit consensus preserving. As  $n \geq 3$ , we may choose an odd-sized subgroup  $M \subseteq N$  containing at least three individuals. (For instance  $M = N$  if  $n$  is odd, or  $M = \{1, 2, 3\}$ .) Define  $F$  as the aggregation rule on  $\mathcal{J}^n$  given by majority voting among  $M$ , i.e.,

$$F(J_1, \dots, J_n) = \{p \in X : |\{i \in M : p \in J_i\}| > |M|/2\} \text{ for all } J_1, \dots, J_n \in \mathcal{J}.$$

I have to show that  $F$  (i) maps into  $\mathcal{J}$ , (ii) is independent, (iii) is implicit consensus preserving, and (iv) is not dictatorial. Properties (ii) and (iv) hold obviously; regarding (ii),  $F$  is in fact even systematic, and regarding (iv) it matters that  $|M| > 1$  and  $|\mathcal{J}| > 1$ . Properties (i) and (iii) both follow as soon as we have shown version 3 of implicit consensus preservation. Consider  $J_1, \dots, J_n \in \mathcal{J}$ . To show that  $F(J_1, \dots, J_n) \in \{J_1, \dots, J_n\}$ , I distinguish between two cases.

*Case 1:*  $X$  is nested, i.e., of the form  $X = \{p_1, \dots, p_m\}^\pm$  where  $m$  is the number of issues and where  $p_1$  entails  $p_2$ ,  $p_2$  entails  $p_3$ , etc. Notice that for each  $J \in \mathcal{J}$  there is a cut-off level  $t = t_J \in \{1, \dots, m+1\}$  such that  $J = \{\neg p_1, \dots, \neg p_{t-1}, p_t, \dots, p_m\}$ , and that

$$F(J_1, \dots, J_n) = J_i = \{\neg p_1, \dots, \neg p_{t_{J_i}-1}, p_{t_{J_i}}, \dots, p_m\},$$

where  $i$  is the *median individual in  $M$* , i.e., the (or an) individual  $i$  in  $M$  such that more than half of the individuals  $j$  in  $M$  have a cut-off level  $t_{J_j} \leq t_{J_i}$ , and more than half of the individuals  $j$  in  $M$  have a cut-off level  $t_{J_j} \geq t_{J_i}$ .

*Case 2:*  $X$  is tiny. As one easily checks, we may assume without loss of generality that  $X$  is redundancy-free and contains only contingent propositions. Then, as  $X$  is tiny, it is either a single-issue agenda or a two-issue agenda. In the first case,  $F(J_1, \dots, J_n)$  is a singleton  $\{p\}$ , which equals  $J_i$  for any individual  $i$  accepting  $p$ . In the second case,  $F(J_1, \dots, J_n)$  is a binary set  $\{p, q\}$ ; since the subgroups  $\{i \in M : p \in J_i\}$  and  $\{i \in M : q \in J_i\}$  each contain a majority of the individuals in  $M$ , these subgroups share at least one individual  $i$ , whose judgment set is therefore  $J_i = \{p, q\} = F(J_1, \dots, J_n)$ . ■

## B.2 Results of Section 4 on aggregation systems

*Proof of Theorem 2.* We consider any aggregation system  $(F_X)_{X \in \mathcal{X}}$ .

1. First, suppose  $(F_X)_{X \in \mathcal{X}}$  is agenda-insensitive.

*Claim 1:*  $(F_X)_{X \in \mathcal{X}}$  is coherent.

Consider  $X, X' \in \mathcal{X}$  with  $X \subseteq X'$  and  $J_1, \dots, J_n \in \mathcal{J}_X$ . Each  $J_i$  is consistent, and thus extendible to a  $J'_i \in \mathcal{J}_{X'}$ . I show that  $F_X(J_1, \dots, J_n) \subseteq F_{X'}(J'_1, \dots, J'_n)$ . Consider any  $p \in F_X(J_1, \dots, J_n)$ . Applying agenda-insensitivity to the agendas  $X$  and  $X'$ , the proposition  $p$  ( $\in X = X \cap X'$ ) and the judgment sets  $J'_i$  ( $\in \mathcal{J}_{X'} = \mathcal{J}_{X \cup X'}$ ), and noting that each  $J'_i$  satisfies  $J'_i \cap X = J_i$  and  $J'_i \cap X' = J'_i$ , we obtain that

$$p \in F_X(J_1, \dots, J_n) \Leftrightarrow p \in F_{X'}(J'_1, \dots, J'_n).$$

So, as  $p \in F_X(J_1, \dots, J_n)$  by assumption,  $p \in F_{X'}(J'_1, \dots, J'_n)$ . QED

*Claim 2:* Each  $F_X$  is independent.

Consider any  $X \in \mathcal{X}$ ,  $p \in X$ , and  $(J_1, \dots, J_n), (J'_1, \dots, J'_n) \in \mathcal{J}_X^n$  such that, for all  $i$ ,  $p \in J_i \Leftrightarrow p \in J'_i$ . Define  $Z$  as the agenda  $\{p, \neg p\} \in \mathcal{X}$ . For each  $i$ , let  $K_i$  be  $\{p\}$  if  $p \in J_i$  (or equivalently,  $p \in J'_i$ ), and as  $\{\neg p\}$  otherwise. Applying agenda-insensitivity to the agendas  $X, Z$  and the judgment sets  $J_i$  ( $\in \mathcal{J}_X = \mathcal{J}_{X \cup X'}$ ), and noting that  $J_i \cap X = J_i$  and  $J_i \cap Z = K_i$ , we obtain

$$p \in F_X(J_1, \dots, J_n) \Leftrightarrow p \in F_Z(K_1, \dots, K_n). \quad (5)$$

Applying agenda-insensitivity again, this time to the agendas  $X, Z$  and the judgment sets  $J'_i$  ( $\in \mathcal{J}_X = \mathcal{J}_{X \cup Z}$ ), and noting that  $J'_i \cap X = J'_i$  and  $J'_i \cap Z = K_i$ , we obtain

$$p \in F_X(J'_1, \dots, J'_n) \Leftrightarrow p \in F_Z(K_1, \dots, K_n). \quad (6)$$

By (5) and (6),  $p \in F_X(J_1, \dots, J_n) \Leftrightarrow p \in F_X(J'_1, \dots, J'_n)$ . QED

2. Now suppose  $(F_X)_{X \in \mathcal{X}}$  is coherent and independent. I prove agenda-insensitivity. Consider any  $X, X' \in \mathcal{X}$ ,  $p \in X \cap X'$  and  $J_1, \dots, J_n \in \mathcal{J}_{X \cup X'}$ , and let us show that  $p \in F_X(J_1 \cap X, \dots, J_n \cap X)$  if and only if  $p \in F_{X'}(J_1 \cap X', \dots, J_n \cap X')$ . Consider the agenda  $Z := \{p, \neg p\} \in \mathcal{X}$ , and for each  $i$  let  $K_i$  be  $\{p\}$  if  $p \in J_i$  and  $\{\neg p\}$  otherwise. Note that  $p \in K_i$  is equivalent to  $p \in J_i \cap X$  and also to  $p \in J_i \cap X'$ , because each of these three statements is equivalent to  $p \in J_i$ . By coherence applied to the agendas  $Z$  and  $X$ , the judgment sets  $K_1, \dots, K_n \in \mathcal{J}_Z$  have extensions  $L_1 \supseteq K_1, \dots, L_n \supseteq K_n$  in  $\mathcal{J}_X$  such that  $F_Z(K_1, \dots, K_n) \subseteq F_X(L_1, \dots, L_n)$ . It follows that

$$p \in F_X(L_1, \dots, L_n) \Leftrightarrow p \in F_Z(K_1, \dots, K_n). \quad (7)$$

Further, for any  $i$ ,  $p \in L_i$  is equivalent to  $p \in K_i$  (as  $L_i \supseteq K_i$ ), which is in turn equivalent to  $p \in J_i \cap X$  (as shown above). Hence, as  $F_X$  is independent,  $p \in F_X(L_1, \dots, L_n) \Leftrightarrow p \in F_X(J_1 \cap X, \dots, J_n \cap X)$ . By (7) it follows that

$$p \in F_X(J_1 \cap X, \dots, J_n \cap X) \Leftrightarrow p \in F_Z(K_1, \dots, K_n). \quad (8)$$

By a similar argument for the agenda  $X'$ ,

$$p \in F_{X'}(J_1 \cap X', \dots, J_n \cap X') \Leftrightarrow p \in F_Z(K_1, \dots, K_n). \quad (9)$$

By (8) and (9),  $p \in F_X(J_1 \cap X, \dots, J_n \cap X) \Leftrightarrow p \in F_{X'}(J_1 \cap X', \dots, J_n \cap X')$ . ■

*Proof of Theorem 3.* Let  $(F_X)_{X \in \mathcal{X}}$  be any aggregation system.

1. First let  $(F_X)_{X \in \mathcal{X}}$  be fully agenda-insensitive.

*Claim 1:*  $(F_X)_{X \in \mathcal{X}}$  is strongly coherent.

Consider  $X, X' \in \mathcal{X}$  with  $X \subseteq \overline{X'}$  and  $J_1, \dots, J_n$  in  $\mathcal{J}_X$ . For any individual  $i$ , since  $J_i$  is consistent, it is extendible to a  $\widehat{J}_i \in \mathcal{J}_{X \cup X'}$ ; we let  $J'_i := \widehat{J}_i \cap X' (\in \mathcal{J}_{X'})$ . We have to show that (i) each  $J'_i$  entails  $J_i$ , and (ii)  $F_{X'}(J'_1, \dots, J'_n)$  entails  $F_X(J_1, \dots, J_n)$ . Regarding (i), for any  $i$ , as  $J'_i \in \mathcal{J}_{X'}$  and  $X \subseteq \overline{X'}$ , the set  $J'_i$  entails exactly one set in  $\mathcal{J}_X$ ; so, by the consistency of  $J'_i \cup J_i (= \widehat{J}_i)$ ,  $J'_i$  entails  $J_i$ . Regarding (ii), consider any  $p \in F_X(J_1, \dots, J_n)$  and let us show that  $F_{X'}(J'_1, \dots, J'_n)$  entails  $p$ . Applying full agenda-insensitivity to the agendas  $X, X'$ , the proposition  $p (\in \overline{X} = \overline{X} \cap \overline{X'})$  and the sets  $\widehat{J}_i$  (which satisfy  $\widehat{J}_i \cap X = J_i$  and  $\widehat{J}_i \cap X' = J'_i$ ), we obtain

$$F_X(J_1, \dots, J_n) \text{ entails } p \Leftrightarrow F_{X'}(J'_1, \dots, J'_n) \text{ entails } p.$$

The left-hand side holds as  $p \in F_X(J_1, \dots, J_n)$ . So,  $F_{X'}(J'_1, \dots, J'_n)$  entails  $p$ . QED

*Claim 2:* Each  $F_X$  is independent on  $\overline{X}$ .

Consider any  $X \in \mathcal{X}$ , any  $p \in \overline{X}$ , and any  $(J_1, \dots, J_n), (J'_1, \dots, J'_n) \in \mathcal{J}_X$  such that, for all  $i$ ,  $J_i$  entails  $p$  if and only if  $J'_i$  does so. Let  $Z$  be the agenda  $\{p, \neg p\} \in \mathcal{X}$ . For each  $i$ , I define  $K_i$  as  $\{p\}$  if  $J_i$  (or equivalently  $J'_i$ ) entails  $p$ , and as  $\{\neg p\}$  otherwise, and I define  $L_i := J_i \cup K_i$  and  $L'_i := J'_i \cup K_i$ . Applying full agenda-insensitivity to the agendas  $X, Z$  and the judgment sets  $L_i$  (which belong to  $\mathcal{J}_{X \cup Z}$  and satisfy  $L_i \cap X = J_i$  and  $L_i \cap Z = K_i$ ), we obtain

$$F_X(J_1, \dots, J_n) \text{ entails } p \Leftrightarrow F_Z(K_1, \dots, K_n) \text{ entails } p. \quad (10)$$

Now applying full agenda-insensitivity to the agendas  $X, Z$  and the judgment sets  $L'_i$  (which belong to  $\mathcal{J}_{X \cup Z}$  and satisfy  $L'_i \cap X = J'_i$  and  $L'_i \cap Z = K_i$ ), we obtain

$$F_X(J'_1, \dots, J'_n) \text{ entails } p \Leftrightarrow F_Z(K_1, \dots, K_n) \text{ entails } p. \quad (11)$$

The relations (10) and (11) jointly imply that  $F_X(J_1, \dots, J_n)$  entails  $p$  if and only if  $F_Z(J'_1, \dots, J'_n)$  entails  $p$ . QED

2. Conversely, assume that  $(F_X)_{X \in \mathcal{X}}$  is strongly coherent and independent on  $\overline{X}$ . To show full agenda-insensitivity, we consider any  $X, X' \in \mathcal{X}$ ,  $p \in \overline{X} \cap \overline{X'}$  and  $J_1, \dots, J_n \in \mathcal{J}_{X \cup X'}$ , and show that  $F_X(J_1 \cap X, \dots, J_n \cap X)$  entails  $p$  if and only if  $F_{X'}(J_1 \cap X', \dots, J_n \cap X')$  entails  $p$ . Consider the agenda  $Z := \{p, \neg p\} \in \mathcal{X}$ . For each  $i$ , define  $K_i$  as  $\{p\}$  if  $J_i$  entails  $p$  and  $\{\neg p\}$  otherwise. By construction,  $J_i$  entails  $K_i$ . So,  $J_i \cap X$  also entails  $K_i$  (as  $J_i \cap X \in \mathcal{J}_X$  and  $p \in \overline{X}$ ), and so  $J_i \cap X$  entails  $p$  if and only if  $p \in K_i$ . For analogous reasons,  $J_i \cap X'$  entails  $K_i$ , and so  $J_i \cap X'$  entails  $p$  if and only if  $p \in K_i$ . By strong coherence applied to the agendas  $Z, X$  (which indeed satisfy  $\overline{Z} \subseteq \overline{X}$  as  $p \in \overline{X}$ ) and the judgment sets  $K_i \in \mathcal{J}_Z$ , there exist some  $L_1, \dots, L_n \in \mathcal{J}_X$  such that each  $L_i$  entails  $K_i$  and  $F_X(L_1, \dots, L_n)$  entails  $F_Z(K_1, \dots, K_n)$ . As  $F_X(L_1, \dots, L_n)$  entails  $F_Z(K_1, \dots, K_n)$  (and as  $F_Z(K_1, \dots, K_n)$  is  $\{p\}$  or  $\{\neg p\}$ ),

$$F_X(L_1, \dots, L_n) \text{ entails } p \Leftrightarrow p \in F_Z(K_1, \dots, K_n). \quad (12)$$

Similarly, for any  $i$ , as  $L_i$  entails  $K_i$  (and as  $K_i$  is  $\{p\}$  or  $\{\neg p\}$ ),  $L_i$  entails  $p$  if and only if  $p \in K_i$ , which was shown to hold if and only if  $J_i \cap X$  entails  $p$ . So, as  $F_X$  is

independent on  $\overline{X}$ ,  $F_X(L_1, \dots, L_n)$  entails  $p$  if and only if  $F_X(J_1 \cap X, \dots, J_n \cap X)$  entails  $p$ . By (12) it follows that

$$F_X(J_1 \cap X, \dots, J_n \cap X) \text{ entails } p \Leftrightarrow p \in F_Z(K_1, \dots, K_n). \quad (13)$$

By an analogous argument for the agenda  $X'$ ,

$$F_{X'}(J_1 \cap X', \dots, J_n \cap X') \text{ entails } p \Leftrightarrow p \in F_Z(K_1, \dots, K_n). \quad (14)$$

The relations (13) and (14) imply that  $F_X(J_1 \cap X, \dots, J_n \cap X)$  entails  $p$  if and only if  $F_{X'}(J_1 \cap X', \dots, J_n \cap X')$  entails  $p$ . ■

*Proof of Theorem 4.* Consider an aggregation system  $(F_X)_{X \in \mathcal{X}}$ .

1. First let this system be focally agenda-insensitive and unanimity preserving. Fix any  $X \in \mathcal{X}$ . Let  $p \in \overline{X}$  be entailed by each of  $J_1, \dots, J_n \in \mathcal{J}_X$ . We have to show that  $F_X(J_1, \dots, J_n)$  entails  $p$ . Consider the agenda  $X' := \{p, \neg p\} \in \mathcal{X}$ . Applying focal agenda-insensitivity to the judgment sets  $J'_i := J_i \cup \{p\} \in \mathcal{J}_{X \cup X'}$  (each of which entails  $p \in \overline{X \cap X'}$ ) and noting that each  $J'_i$  satisfies  $J'_i \cap X = J_i$  and  $J'_i \cap X' = \{p\}$ , we obtain that

$$F_X(J_1, \dots, J_n) \text{ entails } p \Leftrightarrow F_{X'}(\{p\}, \dots, \{p\}) \text{ entails } p.$$

The right-hand side (in which ‘entails’ can be replaced by ‘contains’) holds since  $F_{X'}$  is unanimity preserving. So the left-hand side holds, as desired.

2. Conversely, assume each  $F_X$  is implicit consensus preserving. It then obviously is unanimity preserving. To show focal agenda-insensitivity, we consider any  $X, X' \in \mathcal{X}$  and  $p \in \overline{X \cap X'}$ , and any  $J_1, \dots, J_n \in \mathcal{J}_{X \cup X'}$  each of which entails  $p$ . We show that  $F_X(J_1 \cap X, \dots, J_n \cap X)$  entails  $p$ ; for analogous reasons also  $F_{X'}(J_1 \cap X', \dots, J_n \cap X')$  entails  $p$ , and since both entailments are therefore true they are automatically equivalent, completing the proof. Fix an individual  $i$ . Since  $J_i$  is consistent and entails  $p$ ,  $J_i$  is consistent with  $p$ . So  $J_i \cap X$  is also consistent with  $p$ , and therefore cannot entail  $\neg p$ . Now, as  $p \in \overline{X \cap X'}$ , every judgment set in  $\mathcal{J}_X$  (such as  $J_i \cap X$ ) entails either  $p$  or  $\neg p$ . So  $J_i \cap X$  entails  $p$ . As this is true for all  $i$  and as  $F_X$  is implicit consensus preserving,  $F_X(J_1 \cap X, \dots, J_n \cap X)$  entails  $p$ . ■

### B.3 Results of Appendix A.2 on the scope of agendas

I prove these results in a slightly different order, and draw on additional lemmas.

*Proof of Proposition 2.* The conjunction (disjunction) of elements  $p, q$  of a redundancy-free agenda  $X$  is unique because any two conjunctions (disjunctions) of  $p$  and  $q$  entail each other, hence coincide as  $X$  is redundancy-free. ■

*Proof of Proposition 3.* Suppose an agenda  $X$  is closed under conjunction. Let  $p, q \in X$ . Let  $r \in X$  be the (possibly not unique) conjunction of  $\neg p$  and  $\neg q$ . Then  $\neg r$  is the (possibly not unique) disjunction of  $p$  and  $q$ . Indeed, any  $J \in \mathcal{J}$  contains  $p$  or  $q$  if and only if it is not the case that  $\neg p, \neg q \in J$ ; which is equivalent to  $r \notin J$ , i.e., to  $\neg r \in J$ . Analogously, one can show that if  $X$  is closed under disjunction then any  $p, q \in X$  have a conjunction in  $X$ , namely the proposition  $\neg r$  where  $r$  is a disjunction of  $\neg p$  and  $\neg q$ . ■



*Proof of Proposition 6.* Let  $X$  be a closed redundancy-free agenda and  $\vdash$  the relation of entailment between propositions. The proof proceeds in four claims.

*Claim 1:*  $(X, \vdash)$  is a lattice whose meet and join are given by the operations of conjunction  $\wedge$  and disjunction  $\vee$ , respectively.

First,  $\vdash$  is a partial order: it is clearly reflexive and transitive, and it is also anti-symmetric as  $X$  is redundancy-free. Next, for any  $p, q \in X$ , the conjunction  $p \wedge q$  is the greatest lower bound of  $p$  and  $q$  because, firstly, it is a lower bound (i.e.,  $p \wedge q \vdash p, q$ ), and, secondly, if  $r$  is also a lower bound, then  $r \vdash p \wedge q$ , as  $r \vdash p, q$  and  $\{p, q\} \vdash p \wedge q$ . Analogously, for any  $p \in X$ , the disjunction  $p \vee q$  is the smallest upper bound of  $p$  and  $q$ . QED

*Claim 2:* The lattice  $(X, \vdash)$  is distributive.

Let  $p, q, r \in X$ . Since  $p \vdash p \vee q$  and  $p \vdash p \vee r$ , we have (\*)  $p \vdash (p \vee q) \wedge (p \vee r)$ . Since  $q \wedge r$  entails  $q$  (which entails  $p \vee q$ ) and entails  $r$  (which entails  $p \vee r$ ), (\*\*)  $q \wedge r \vdash (p \vee q) \wedge (p \vee r)$ . By (\*) and (\*\*),

$$p \vee (q \wedge r) \vdash (p \vee q) \wedge (p \vee r). \quad (15)$$

We next show the converse implication,

$$(p \vee q) \wedge (p \vee r) \vdash p \vee (q \wedge r). \quad (16)$$

Consider any  $J \in \mathcal{J}$  containing  $(p \vee q) \wedge (p \vee r)$ , and let us show that  $p \vee (q \wedge r) \in J$ . As  $(p \vee q) \wedge (p \vee r)$  entails  $p \vee q$  and also  $p \vee r$ , we have  $p \vee q, p \vee r \in J$ . So,  $J$  contains  $p$  or  $q$  (or both), and contains  $p$  or  $r$  (or both). So,  $J$  contains  $p$  or contains both  $q$  and  $r$ ; in the latter case,  $q \wedge r \in J$ . Since, as we have shown,  $p \in J$  or  $q \wedge r \in J$ , we have  $p \vee (q \wedge r) \in J$ , as desired. By (15) and (16), and by the asymmetry of  $\vdash$ ,  $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$ . By analogous arguments,  $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$ . QED

*Claim 3:*  $X$  has a smallest element  $\perp$  and a greatest element  $\top$ , namely the contradiction  $\bigwedge_{p \in X} p$  and the tautology  $\bigvee_{p \in X} p$ , respectively.

It is obvious that  $\bigwedge_{p \in X} p$  entails each  $q \in X$  and that each  $q \in X$  entails  $\bigvee_{p \in X} p$ . QED

*Claim 4:* For each  $p \in X$ ,  $p \wedge \neg p = \perp$  and  $p \vee \neg p = \top$  (i.e.,  $\neg p$  is the algebraic complement of  $p$ ).

Let  $p \in X$ . Since  $\{p, \neg p\}$  is inconsistent,  $p \wedge \neg p = \perp$ . Since every  $J \in \mathcal{J}$  contains  $p$  or  $\neg p$ , every  $J \in \mathcal{J}$  contains  $p \vee \neg p$ , whence  $p \vee \neg p = \top$ . ■

**Lemma 5** *The notions of consistency, entailment, conjunction and disjunction are preserved by any extension of the agenda (and thus can be used without referring explicitly to an agenda). Formally, for any agenda  $X$  and any superagenda  $X'$  (e.g., the scope of  $X$ ),*

- (a) *a set  $S \subseteq X$  is consistent w.r.t.  $X$  if and only if it is so w.r.t.  $X'$ ,*
- (b) *a proposition  $p \in X$  (or set  $S \subseteq X$ ) entails a proposition  $p' \in X$  (or set  $S' \subseteq X$ ) w.r.t.  $X$  if and only if it does so w.r.t.  $X'$ ,*
- (c) *a proposition  $r \in X$  is the (or a) conjunction/disjunction of certain propositions in  $X$  w.r.t.  $X$  if and only if it is so w.r.t.  $X'$ .*

*Proof.* Part (b) follows from part (a), since the entailment notion is reducible to the consistency notion (e.g.,  $p$  entails  $p'$  if and only if  $\{p, \neg p'\}$  is inconsistent). Further, part (c) follows from part (b), since the notions of conjunction and disjunction are reducible to the entailment notion:  $r$  is a conjunction of a set of propositions  $S$  if and only if  $r$  and  $S$  entail each other, and  $r$  is a disjunction of the set of propositions  $S$  if and only if  $\neg r$  and  $\{\neg p : p \in S\}$  entail each other. To prove part (a), recall that (\*)  $\mathcal{J}_X = \{J' \cap X : J' \in \mathcal{J}_{X'}\}$ . Consider any  $S \subseteq X$ . First, let  $S$  be consistent w.r.t.  $X$ . Then there is a  $J \in \mathcal{J}_X$  such that  $S \subseteq J$ . By (\*), we may write  $J = J' \cap X$  for some  $J' \in \mathcal{J}_{X'}$ . Clearly,  $S \subseteq J'$ , whence  $S$  is consistent w.r.t.  $X'$ . Conversely, assume  $S$  is consistent w.r.t.  $X'$ . Then we may choose a  $J' \in \mathcal{J}_{X'}$  such that  $S \subseteq J'$ . By (\*),  $\mathcal{J}_X$  contains  $J := J' \cap X$ . Note that  $S \subseteq J$ . So  $S$  is consistent w.r.t.  $X$ . ■

**Lemma 6** *For any agenda  $X$  and any closed (redundancy-free) superagenda  $X'$  – possibly  $X$  itself or the scope of  $X$  – a set  $A \subseteq X$  is consistent if and only if, in  $X'$ ,  $\bigwedge_{p \in A} p \neq \perp$ .*

*Proof.* Let  $X$  and  $X'$  be as specified. By Lemma 5, we need not distinguish between consistency w.r.t.  $X$  and w.r.t.  $X'$ . We proceed by showing three claims.

*Claim 1:*  $\perp$  is the only element of  $X'$  which is not contained in any rational judgment set  $J \in \mathcal{J}_{X'}$ .

This follows from four facts (some of which draw on Proposition 6): (i)  $\perp$  is the only element of  $X'$  which entails its own algebraic complement (a basic fact about Boolean algebras); (ii) the algebraic complement of an element  $p$  is its (agenda-theoretic) negation  $\neg p$ ; (iii) an element  $p$  entails another  $q$  if and only if no  $J \in \mathcal{J}_{X'}$  contains both  $p$  and  $\neg q$ ; (iv) every  $J \in \mathcal{J}_{X'}$  contains exactly one of member of each pair  $p, \neg p \in X$ . QED

*Claim 2:* For any  $J \in \mathcal{J}_{X'}$  and any  $A \subseteq J$ , we have  $\bigwedge_{p \in A} p \in J$ .

Let  $J \in \mathcal{J}_{X'}$  and  $A \subseteq J$ . By Proposition 6 we can think of ‘ $\wedge$ ’ alternatively as the conjunction operator (defined agenda-theoretically) or the meet (defined Boolean-algebraically). The claim holds by induction on the size of  $A$ . If  $A = \emptyset$ , the claim holds because then  $\bigwedge_{p \in A} p = \top$  and  $\top \in J$  (as  $\top = \neg \perp$ , where  $\perp \notin J$  by Claim 1). Now assume  $A$  has size  $m \geq 1$  and suppose the claim holds for any smaller size. We may write  $A = A' \cup \{q\}$  with  $q \notin A'$ . By induction hypothesis,  $\bigwedge_{p \in A'} p \in J$ . Since  $J$  contains both  $\bigwedge_{p \in A'} p$  and  $q$ ,  $J$  contains their conjunction  $(\bigwedge_{p \in A'} p) \wedge q = \bigwedge_{p \in A} p$  by definition of conjunction. QED

*Claim 3:* A set  $A \subseteq X'$  is consistent if and only if  $\bigwedge_{p \in A} p \neq \perp$ .

First, let  $A \subseteq X'$  be consistent. Then it has an extension  $J \in \mathcal{J}_{X'}$ , which by Claim 2 contains  $\bigwedge_{p \in A} p$ . So by Claim 1  $\bigwedge_{p \in A} p \neq \perp$ . Conversely, assume  $\bigwedge_{p \in A} p \neq \perp$ . Then by Claim 1 there is a  $J \in \mathcal{J}_{X'}$  containing  $\bigwedge_{p \in A} p$ . So, as  $\bigwedge_{p \in A} p$  entails each  $p \in A$ ,  $J$  contains each  $p \in A$ , i.e.,  $A \subseteq J$ . ■

*Proof of Proposition 4.* Let  $X$  be an agenda. As one easily checks, we may assume

without loss of generality. that  $X$  is redundancy-free.

1. In this part we show that we may assume without loss of generality. that  $X$  is a ‘semantic’ agenda given as follows: there exists a finite set of ‘worlds’  $\Omega \neq \emptyset$  such that  $X \subseteq 2^\Omega$ , where (i) each issue takes the form  $\{A, \overline{A}\}$  (I write  $\overline{A}$  for the complement  $\Omega \setminus A$  of any set  $A \subseteq \Omega$ ), (ii) the set  $\mathcal{J}_X$  of rational judgment sets consists of those sets  $J \subseteq X$  which contain exactly one member of each issue and satisfy  $\bigcap_{A \in J} A \neq \emptyset$ , and (iii) rational judgment sets in  $\mathcal{J}_X$  correspond to worlds in  $\Omega$ , in the sense that the assignment  $J \mapsto \bigcap_{A \in J} A$  defines a bijection from  $\mathcal{J}_X$  to  $\{\{\omega\} : \omega \in \Omega\}$ .

To show this, we consider any agenda  $V$  and construct a semantic agenda  $X$  of the given sort to which  $V$  is isomorphic. Let the set of worlds be  $\Omega := \mathcal{J}_V$ . To each  $p \in V$  corresponds a set of worlds, the ‘extension’ of  $p$ , given by  $[p] := \{\omega \in \Omega : p \in \omega\}$ . Note that the assignment  $p \mapsto [p]$  defines a bijection from  $V$  to the set  $X := \{[p] : p \in V\}$ . I define an agenda by the set  $X$ , endowed with

- issues defined as the sets  $\{[p], [\neg p]\}$  (which indeed partition  $X$  into pairs, since the sets  $\{p, \neg p\}$  partition  $V$  into pairs and since  $p \mapsto [p]$  maps  $V$  bijectively to  $X$ ),
- rational judgment sets defined as the sets  $J \subseteq X$  containing exactly one member of each issue and satisfying  $\bigcap_{A \in J} A \neq \emptyset$ .

This agenda  $X$  satisfies (i) since  $[\neg p] = \overline{[p]}$  for all  $p \in V$ , and satisfies (ii) immediately by definition. To show that it satisfies (iii), we first show that for each  $J \in \mathcal{J}_X$  the intersection  $\bigcap_{A \in J} A$  ( $\neq \emptyset$ ) is indeed a singleton. Assume for a contradiction that it contains distinct  $\omega, \omega' \in \Omega$ . Since  $\omega \neq \omega'$ , there is a  $p \in V$  such that  $p \in \omega' \setminus \omega$  and  $\neg p \in \omega \setminus \omega'$ . So,  $\omega \notin [p]$  and  $\omega' \notin [\neg p]$ . Since  $J$  contains either  $[p]$  or  $[\neg p]$ , it follows that either  $\omega \notin \bigcap_{A \in J} A$  or  $\omega' \notin \bigcap_{A \in J} A$ , a contradiction. Second, one has to check injectivity and surjectivity of the mapping from  $\mathcal{J}_X$  to  $\{\{\omega\} : \omega \in \Omega\}$ ; we leave this to the reader.

Finally, to show that  $V$  and  $X$  are isomorphic (as agendas), it suffices to show that  $p \mapsto [p]$  defines an (agenda) isomorphism. This is so because the assignment  $p \mapsto [p]$  is bijective, and bijectively maps the issues  $\{p, \neg p\}$  of  $V$  to those of  $X$ , and the rational judgment sets of  $V$  to those of  $X$  (the latter can be shown by verifying that the assignment  $J \mapsto \{[p] : p \in J\}$  defines a bijection from  $\mathcal{J}_V$  to  $\mathcal{J}_X$ ).

2. From now on we assume that  $X$  takes the semantic form defined in part 1. In the current part, we show the existence claim. As one can check,  $X$  is a subagenda of the agenda  $X' := 2^\Omega$  whose issues are the pairs  $\{A, \overline{A}\}$  ( $A \subseteq \Omega$ ) and whose rational judgment sets are the sets of the form  $\{A \subseteq \Omega : \omega \in A\}$  ( $\omega \in \Omega$ ). It suffices to show that  $X'$  is a *minimal closed* extension of  $X$ . First,  $X'$  is closed, where the conjunction is given by the intersection, and the disjunction by the union. Second, we have to show minimality. Consider any superagenda  $X''$  of  $X$  which is a strict subagenda of  $X'$ . We have to show that  $X''$  is not closed. As  $X''$  is a subagenda of  $X'$ , it inherits its issues from  $X'$ , and thus  $X''$  is closed under complement:  $A \in X'' \Rightarrow \overline{A} \in X''$ . Since  $X' (= 2^\Omega)$  is the *only* subset of  $2^\Omega$  which includes  $X$  and is closed under intersection and complement, and since  $X''$  is closed under complement,  $X''$  cannot be closed under intersection. It follows that  $X''$  is not closed (i.e., not closed under conjunction), by the following argument. Choose any  $A, B \in X''$  such that  $A \cap B \notin X''$ . Suppose for a contradiction that  $X''$  contains a  $C$  which (relative to agenda  $X''$ ) is the conjunction

of  $A$  and  $B$ , i.e., is equivalent to  $\{A, B\}$ . Since  $A \cap B \notin X''$ ,  $C \neq A \cap B$ . So, since also  $C \subseteq A$  and  $C \subseteq B$  (as  $C$  entails  $A$  and  $B$  relative to the agenda  $X''$ ), we have  $C \subsetneq A \cap B$ . Choose any  $\omega \in (A \cap B) \setminus C$ . Note that  $J'' := \{D \in X'' : \omega \in D\}$  belongs to  $\mathcal{J}_{X''}$ , and contains  $A$  and  $B$  but not  $C$ . So (still relative to agenda  $X''$ )  $\{A, B\}$  does not entail  $C$ , a contradiction since  $C$  is the conjunction of  $A$  and  $B$ .

3. Finally, we show the uniqueness claim. Since the agenda  $X'$  defined in part 2 is a minimal closed extension of  $X$ , it suffices to show that any other such extension of  $X$  is equal to  $X'$  up to relabelling. Let  $Z$  be an arbitrary minimal closed superagenda of  $X$ . We need to define an agenda isomorphism  $f : X' \rightarrow Z$  which is constant on  $X$ . For all  $\omega \in \Omega$  and all  $Y \subseteq X' (= 2^\Omega)$ , let  $Y_\omega := \{A \in Y : \omega \in A\}$ , and for all  $B \in X' (= 2^\Omega)$  let

$$p_B := \bigvee_{\omega \in B} \left( \bigwedge X_\omega \right) \quad (\in Z). \quad (17)$$

Here and in what follows, let ' $\vee$ ', ' $\wedge$ ' and ' $\neg$ ' refer to the disjunction, conjunction and negation operators of  $Z$  (rather than of  $X$  or  $X'$ ). By Proposition 6, ' $\vee$ ', ' $\wedge$ ' and ' $\neg$ ' can alternatively be viewed as the algebraic operations of join, meet and complement in the Boolean algebra  $Z$ . So, standard algebraic rules apply, such as associativity, commutativity and distributivity of  $\vee$  and  $\wedge$ . Also, let  $\top$  and  $\perp$  be the greatest and smallest elements of the Boolean algebra  $Z$ , respectively; clearly,  $\top$  is the (only) tautology and  $\perp$  the (only) contradiction of the agenda  $Z$ .

*Claim 1:* For all  $Y \subseteq X$ ,  $A \in X \setminus Y$  and  $\omega \in \Omega$  we write  $Y_\omega^A := Y_\omega \cup \{A\}$ . For every subagenda  $Y$  of  $X$ ,  $A \in X \setminus Y$ , and  $\omega \in \overline{A}$ , either  $\bigwedge Y_\omega^A = \perp$  or  $Y_\omega^A = Y_{\omega'}^A$  for some  $\omega' \in A$ .

Consider any subagenda  $Y$  of  $X$ ,  $A \in X \setminus Y$ , and  $\omega \in \overline{A}$ . First assume  $Y_\omega^A$  is inconsistent w.r.t. agenda  $X$ . Then  $\bigwedge Y_\omega^A = \perp$  by Lemma 6. Now assume  $Y_\omega^A$  is consistent w.r.t. agenda  $X$ . So there is an  $\omega' \in \cap Y_\omega^A$ . In particular,  $\omega' \in \cap Y_\omega$ . So, for each  $B \in Y$ ,  $\omega \in B \Rightarrow \omega' \in B$ . In fact, the ' $\Rightarrow$ ' can be replaced by ' $\Leftrightarrow$ ', since  $\omega$  and  $\omega'$  belong to the same number of sets  $B$  in  $Y$  (i.e., to half the these sets, as  $B \in Y \Leftrightarrow \overline{B} \in Y$ ). So,  $Y_\omega = Y_{\omega'}$ , and hence,  $Y_\omega^A = Y_{\omega'}^A$ . QED

*Claim 2:* For all  $B \in X'$ ,  $\neg p_B = p_{\overline{B}}$ .

Let  $B \in X'$ . Since  $\neg$  coincides with the algebraic complement operation in  $Z$ , it suffices to show that  $p_B \vee p_{\overline{B}} = \top$  and  $p_B \wedge p_{\overline{B}} = \perp$ .

We first prove that  $p_B \vee p_{\overline{B}} = \top$ . Since

$$p_B \vee p_{\overline{B}} = \left[ \bigvee_{\omega \in B} \bigwedge X_\omega \right] \vee \left[ \bigvee_{\omega \in \overline{B}} \bigwedge X_\omega \right] = \bigvee_{\omega \in \Omega} \bigwedge X_\omega,$$

we have to prove that  $\bigvee_{\omega \in \Omega} \bigwedge X_\omega = \top$ . We first show that

$$\bigvee_{\omega \in \Omega} \bigwedge X_\omega = \bigvee_{\omega \in \Omega} \bigwedge Y_\omega, \quad (18)$$

where  $Y$  is any set of the form  $X \setminus \{A, \bar{A}\}$  with  $A \in X$ . Note that

$$\begin{aligned} \bigvee_{\omega \in \Omega} \bigwedge X_\omega &= \left[ \bigvee_{\omega \in A} \bigwedge X_\omega \right] \vee \left[ \bigvee_{\omega \in \bar{A}} \bigwedge X_\omega \right] \\ &= \left[ \bigvee_{\omega \in A} \bigwedge Y_\omega^A \right] \vee \left[ \bigvee_{\omega \in \bar{A}} \bigwedge Y_\omega^{\bar{A}} \right], \end{aligned}$$

where the last expression uses notation introduced in Claim 1. This expression is a disjunction of terms (disjuncts) of two types: any  $\bigwedge Y_\omega^A$  with  $\omega \in A$  (type 1) and any  $\bigwedge Y_\omega^{\bar{A}}$  with  $\omega \in \bar{A}$  (type 2). The result is not affected by adding the following new disjuncts: any  $\bigwedge Y_\omega^A$  with  $\omega \in \bar{A}$  (type 3) and any  $\bigwedge Y_\omega^{\bar{A}}$  with  $\omega \in A$  (type 4). Indeed, by Claim 1 each new disjunct of type 3 is either  $\perp$  or coincides with a disjunct of type 1, and any new disjunct of type 4 is either  $\perp$  or coincides with a disjunct of type 2. After adding these new disjuncts and re-grouping, the expression becomes

$$\left[ \bigvee_{\omega \in \Omega} \bigwedge Y_\omega^A \right] \vee \left[ \bigvee_{\omega \in \Omega} \bigwedge Y_\omega^{\bar{A}} \right].$$

Noting that each  $Y_\omega^A$  equals  $\{A\} \cup Y_\omega$  and each  $Y_\omega^{\bar{A}}$  equals  $\{\bar{A}\} \cup Y_\omega$ , and then using distributivity twice, the last expression reduces to

$$\begin{aligned} &\left[ A \bigwedge \left( \bigvee_{\omega \in \Omega} \bigwedge Y_\omega \right) \right] \vee \left[ \bar{A} \bigwedge \left( \bigvee_{\omega \in \Omega} \bigwedge Y_\omega \right) \right] \\ &= \left[ A \vee \bar{A} \right] \bigwedge \left( \bigvee_{\omega \in \Omega} \bigwedge Y_\omega \right) \\ &= \bigvee_{\omega \in \Omega} \bigwedge Y_\omega. \end{aligned}$$

This proves (18). By an analogous argument, one can show that (unless  $Y = \emptyset$ ), we have  $\bigvee_{\omega \in \Omega} \bigwedge Y_\omega = \bigvee_{\omega \in \Omega} \bigwedge Y'_\omega$  for a set  $Y'$  of the form  $Y \setminus \{A, \bar{A}\}$  with  $A \in Y$ ; which together with (18) yields that  $\bigvee_{\omega \in \Omega} \bigwedge X_\omega = \bigvee_{\omega \in \Omega} \bigwedge Y'_\omega$ . Continuing in this fashion, we ultimately obtain that  $\bigvee_{\omega \in \Omega} \bigwedge X_\omega = \bigvee_{\omega \in \Omega} \bigwedge \emptyset_\omega = \top$ , as desired.

We finally have to prove that  $p_B \wedge p_{\bar{B}} = \perp$ . Using distributivity twice,

$$\begin{aligned} p_B \wedge p_{\bar{B}} &= \left[ \bigvee_{\omega \in B} \bigwedge X_\omega \right] \wedge \left[ \bigvee_{\omega' \in \bar{B}} \bigwedge X_{\omega'} \right] \\ &= \bigvee_{\omega \in B} \left( \left[ \bigwedge X_\omega \right] \wedge \left[ \bigvee_{\omega' \in \bar{B}} \bigwedge X_{\omega'} \right] \right) \\ &= \bigvee_{\omega \in B} \left( \bigvee_{\omega' \in \bar{B}} \left( \left[ \bigwedge X_\omega \right] \wedge \left[ \bigwedge X_{\omega'} \right] \right) \right). \end{aligned}$$

It thus suffices to show that for all  $\omega \in B$  and  $\omega' \in \overline{B}$  we have  $[\bigwedge X_\omega] \wedge [\bigwedge X_{\omega'}] = \perp$ . Let  $\omega \in B$  and  $\omega' \in \overline{B}$ . Clearly  $\omega \neq \omega'$ , and so there is an  $A \in X$  such that  $\omega \in A$  and  $\omega' \in \overline{A}$ . Since  $A \in X_\omega$ ,  $\bigwedge X_\omega$  entails  $A$ . Analogously, since  $\overline{A} \in X_{\omega'}$ ,  $\bigwedge X_{\omega'}$  entails  $\overline{A}$ . It follows that  $[\bigwedge X_\omega] \wedge [\bigwedge X_{\omega'}]$  entails  $A \wedge \overline{A}$ . As  $A \wedge \overline{A} = \perp$  (since  $A$  and  $\overline{A}$  are complements in the algebra  $Z$ ), it follows that  $[\bigwedge X_\omega] \wedge [\bigwedge X_{\omega'}]$  entails  $\perp$ , hence, equals  $\perp$ . QED

*Claim 3:*  $p_B = B$  for all  $B \in X$ .

Consider any  $B \in X$ . We regard  $B$  as an element of the extended agenda  $Z \supseteq X$ . Since  $Z$  is redundancy-free, it suffices to show that  $p_B$  and  $B$  entail each other. We first show that  $p_B$  entails  $B$ . Since  $p_B$  is the *least* upper bound of all  $\bigwedge X_\omega$  with  $\omega \in B$ , it suffices to show that  $B$  is an upper bound, i.e., that each of these  $\bigwedge X_\omega$  entails  $B$ . This is so because for each  $\omega \in B$  the set  $X_\omega$  contains  $B$ . Second, we show that  $B$  entails  $p_B$ , or equivalently, that  $\neg p_B$  entails  $\neg B$ . This follows from the previous argument applied to  $\overline{B}$  rather than  $B$ , because  $\neg p_B = p_{\overline{B}}$  by Claim 2 and because  $\neg B = \overline{B}$  (as  $Z$  is a superagenda of  $X$ , so that  $B$ 's  $Z$ -relative negation  $\neg p_B$  coincides with  $B$ 's  $X$ -relative negation  $\overline{B}$ ). QED

*Claim 4:* For all  $A, B \in X'$ ,  $p_{A \cup B} = p_A \vee p_B$  and  $p_{A \cap B} = p_A \wedge p_B$ .

Let  $A, B \in X'$ . The first identity holds immediately by definition of  $p_A$  and  $p_B$ . As for the second identity, using de Morgan's Law (valid in Boolean algebras) and then Claim 2,  $p_A \wedge p_B = \neg(\neg p_A \vee \neg p_B) = \neg(p_{\overline{A}} \vee p_{\overline{B}})$ . Now using the first identity, it follows that  $p_{A \cap B} = \neg p_{\overline{A} \cup \overline{B}}$ , which reduces to  $p_{A \cap B}$  by  $\overline{A} \cup \overline{B} = \overline{A \cap B}$  and Claim 2. QED

*Claim 5:*  $Z = \{p_B : B \in X'\}$ .

The set  $S := \{p_B : B \in X'\} (\subseteq Z)$  is closed under negation by Claim 2, hence defines a subagenda of  $Z$ . The agenda  $S$  is closed because for any  $B, C \in X'$  the disjunction of  $p_B$  and  $p_C$  (relative to the agenda  $Z$ ) equals  $p_{B \cup C}$  by Claim 4, hence belongs to the agenda  $S$  (relative to which it of course still defines the disjunction of  $p_B$  and  $p_C$ ). Moreover, the agenda  $S$  includes  $X$  by Claim 3, hence is a superagenda of  $X$ . Since  $Z$  is by definition a *minimal* closed superagenda of  $X$ , it follows that  $S = Z$ . QED

*Claim 6:* For all  $A, B \in X'$ ,  $A \subseteq B$  if and only if  $p_A$  entails  $p_B$ .

For each  $\omega \in \Omega$  we have  $p_{\{\omega\}} \neq \perp$ ; this is because the set  $X_\omega$  is consistent with respect to agenda  $X$ , and hence  $p_{\{\omega\}} = \bigwedge X_\omega \neq \perp$  by Lemma 6. Now consider any  $A, B \in X'$ . First, if  $A \subseteq B$ , then  $p_A$  clearly entails  $p_B$  since  $p_B$  is a disjunction of *at least* those terms of which  $p_A$  is a disjunction. Conversely, now assume that  $p_A$  entails  $p_B$ . As  $A \setminus B \subseteq \overline{B}$ ,  $p_{A \setminus B}$  entails  $p_{\overline{B}}$ ; and so, as  $p_{\overline{B}} = \neg p_B$  by Claim 2,  $p_{A \setminus B}$  entails  $\neg p_B$ . Also, as  $A \setminus B \subseteq A$ ,  $p_{A \setminus B}$  entails  $p_A$ ; and so, as  $p_A$  entails  $p_B$ ,  $p_{A \setminus B}$  entails  $p_B$ . Since, as we have shown,  $p_{A \setminus B}$  entails both  $\neg p_B$  and  $p_B$ , it entails  $\neg p_B \wedge p_B = \perp$ . Hence,  $p_{A \setminus B} = \perp$ . It follows that  $A \setminus B = \emptyset$ , i.e.,  $A \subseteq B$ , since if there were an  $\omega \in A \setminus B$ , then  $p_{\{\omega\}}$  would entail  $p_{A \setminus B}$ , whence  $p_{\{\omega\}} = \perp$ , in contradiction with what was shown at the start of the proof of the claim. QED

*Claim 7:* The assignment  $B \mapsto p_B$  defines an agenda isomorphism between  $X'$  and  $Z$  which is constant on  $X$ . (This completes the proof.)

This assignment – call it  $f$  – is constant on  $X$  by Claim 3, and surjective by Claim 4. To show injectivity, consider distinct  $A, B \in X'$ . We may assume without loss of generality that  $A \not\subseteq B$  (since otherwise the roles of  $A$  and  $B$  can be interchanged). By Claim 6,  $p_A$  does not entail  $p_B$ , and so  $p_A \neq p_B$ .

It remains to show that  $f$  preserves the agenda structure: the issues (resp. negation operator) and the interconnections. This could be deduced from Claims 2, 5 and 6 since, firstly, by these claims the (bijective) function  $f$  is a Boolean-algebra isomorphism, and, secondly, for a closed agenda, the agenda structure and the Boolean-algebra structure are interdefinable, as can be verified; see Proposition 6.<sup>33</sup> But let me give a direct proof. First,  $f$  preserves the issues structure, since for each  $A \in X'$  we have  $\neg p_A = p_{\overline{A}}$  (by Claim 2) and  $\overline{A}$  is the  $X'$ -relative negation of  $A$ . Second, consider a set  $S \subseteq X'$ ; we show that  $S$  is consistent (in the sense of  $X'$ ) if and only if its image  $\{p_B : B \in S\}$  is consistent (in the sense of  $Z$ ). This holds for the following reasons.  $S$  is consistent if and only if  $\cap S \neq \emptyset$ , which is in turn equivalent to  $p_{\cap S} \neq p_{\emptyset}$ , i.e., to  $p_{\cap S} \neq \perp$ . By Claim 4, the latter is equivalent to  $\bigwedge_{B \in S} p_B \neq \perp$ , which is in turn equivalent to the consistency of  $\{p_B : B \in S\}$  by Lemma 6. ■

*Proof of Lemma 1.* This lemma follows from the proof of Proposition 4. ■

*Proof of Proposition 5.* Let  $X$  be an agenda. It suffices to show that for each  $J \in \mathcal{J}_X$  and  $p \in \overline{X}$ ,  $J$  entails  $p$  or entails  $\neg p$ , or equivalently,  $\bigwedge_{q \in J} q$  entails  $p$  or entails  $\neg p$ . This follows from the fact that, by Lemma 1,  $\bigwedge_{q \in J} q$  is an atom of  $\overline{X}$ , i.e., a logically strongest element of  $\overline{X} \setminus \{\perp\}$ . ■

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<sup>33</sup>For instance, a subset  $A$  is consistent in the agenda sense if and only if its algebraic meet is not  $\perp$ , by Lemma 6.